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Computing quadric surface intersections based on an analysis of plane cubic curves ☆

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Abstract

Computing the intersection curve of two quadrics is a fundamental problem in computer graphics and solid modeling. We present an algebraic method for classifying and parameterizing the intersection curve of two quadric surfaces. The method is based on the observation that the intersection curve of two quadrics is birationally related to a plane cubic curve. In the method this plane cubic curve is computed first and the intersection curve of the two quadrics is then found by transforming the cubic curve by a rational quadratic mapping. Topological classification and parameterization of the intersection curve are achieved by invoking results from algebraic geometry on plane cubic curves.

1. Introduction

Quadric surfaces, or surfaces of degree two, are the simplest curved surfaces and they are widely used in computer graphics and solid modeling systems [6,8,27, 28,31,41,42]. In these applications it is often necessary to compute the intersection of two or more quadric surfaces [10,12,14,18,22,40]. For brevity, following [19],

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the intersection curve of two quadrics will be referred to as a QSIC (quadric surface intersection curve). The work reported here is about an algebraic method to compute QSICs based on a birational mapping between QSICs and plane cubic curves.

1.1. Outline of the method

We propose an algebraic method for computing the QSIC, based on the result that the intersection curve of two quadrics is birationally related to a plane cubic curve [4,35]. A QSIC, which is a quartic space curve of the first species, is projected into a plane cubic curve through a point $\mathbf{X}_0$ on the QSIC. This projection is a one-to-one birational transformation $\mathcal{P}$ between a plane not passing through $\mathbf{X}_0$ and one of the two quadric surfaces under consideration; it is, in fact, a generalized stereographic projection. With this projection the computation of a QSIC is facilitated by a reduction of both its algebraic degree (from degree 4 to degree 3) and its dimension (from 3D to 2D).

Since our goal is to compute the QSIC of two given quadrics, the projection of the QSIC, which is a plane cubic curve, must be found first. After the cubic curve is processed, the QSIC is obtained by mapping it onto one of the two given quadric surfaces.

To devise an algorithm based on the above scheme, we are forced to address a number of computational issues as well as study the ramifications of this scheme in all cases. As a result, we have obtained an algorithm that is simple to implement and capable of classifying a general QSIC and computing its parameterization.

As our method relies on a birational mapping between an algebraic plane curve and the intersection curve of two algebraic surfaces, it is similar to the approaches in [3,12,13], but with one important difference: in our method the QSIC is mapped into a plane cubic curve, while a plane quartic curve is utilized in the other methods. Hence, in this sense our method can be regarded as an improvement over those other methods.

Some conventions on notation are in order. Scalar values and scalar functions are denoted by lowercase and uppercase letters, respectively. Boldface uppercase and lowercase letters denote 3D and 2D points or their coordinate vectors, respectively. Matrices are also denoted by boldface uppercase letters. Lines, surfaces, sets and intervals are denoted by calligraphic uppercase letters. For example, a quadric surface is denoted by $\mathcal{S} : F(\mathbf{X}) \equiv \mathbf{X}\mathbf{A}\mathbf{X}^{\mathrm{T}} = 0$, where $\mathbf{X}$ is a 4D row vector of homogeneous coordinates and $\mathbf{A}$ a 4×4 symmetric matrix. The coefficients of all quadric surfaces treated in this paper are real numbers, and the ground field of computation is the field of real numbers unless specified otherwise.

The remainder of this paper is organized as follows. We first review previous work in Section 1.2. In Section 2 we present some preliminaries; these include rational quadratic parameterizations of quadrics and classifications of plane cubic curves and QSICs. In Section 3 plane cubic curves birationally related to QSICs are derived. The intersection algorithm is presented in Section 4, followed by discussions relating how to compute a QSIC from its corresponding cubic curve and how to classify and parameterize a QSIC. In Section 5 several examples are presented to illustrate major steps of the algorithm in a variety of cases. The paper concludes in Section 6 with a summary of our results and a few open questions for future research.

1.2. Relevant prior work

A well-known algebraic method for computing QSICs is due to Levin [18,19]. Let $\mathcal{S}_0 : \mathbf{XAX}^{\mathrm{T}} = 0$ and $\mathcal{S}_1 : \mathbf{XBX}^{\mathrm{T}} = 0$ be two distinct quadrics, where $\mathbf{A}$ and $\mathbf{B}$ are 4×4 real symmetric matrices, and $\mathbf{X}$ is a row vector of homogeneous coordinates. Levin's method is based on the observation that there always exists a ruled quadric in the pencil $\mathbf{X}(\lambda\mathbf{A} + \mu\mathbf{B})\mathbf{X}^{\mathrm{T}} = 0$ spanned by $\mathcal{S}_0$ and $\mathcal{S}_1$. This ruled quadric, called a "parameterization surface," is parameterized as

$$\mathbf{S}(u, v) = \tilde{\mathbf{R}}(u) + v\tilde{\mathbf{T}}(u),$$

where $\tilde{\mathbf{R}}(u)$ is the base curve of the ruling and $\tilde{\mathbf{T}}(u)$ is the direction vector of the generating line passing through $\tilde{\mathbf{R}}(u)$. Substituting $\mathbf{S}(u, u)$ into either $\mathcal{S}_0$ or $\mathcal{S}_1$, one can solve for v in terms of u to obtain a parameterization of the QSIC of $\mathcal{S}_0$ and $\mathcal{S}_1$ in the form,

$$\mathbf{Q}(u) = \mathbf{R}(u) \pm \sqrt{D(u)}\mathbf{T}(u). \tag{1}$$

Only those values of u for which $D(u) \geqslant 0$ give rise to real points on the QSIC of $\mathcal{S}_0$ and $\mathcal{S}_1$. By identifying the intervals over which $D(u) \geqslant 0$, a segmentation of the QSIC can be derived.

Levin's method is useful mainly for tracing and rendering a QSIC, rather than for classification. In some cases the parameterization computed with this method may not be appropriate. For instance, when a QSIC is singular or reducible, the method still generates a parameterization involving a square root, though in this case a rational parameterization for the QSIC is available. Levin's method was later refined and implemented in GMSOLID by Sarraga [31] and also revised and extended in [40].

By studying the eigenvalues and the generalized eigenspaces of the system $\mathbf{AB} - \lambda\mathbf{I}$ under the mild assumption that $\mathbf{A}^2 = \mathbf{I}$, Ocken et al. show [24] that two quadrics $\mathbf{XAX}^{\mathrm{T}} = 0$ and $\mathbf{XBX}^{\mathrm{T}} = 0$ can be converted simultaneously by a projective transformation into two canonical forms whose intersection curve can be determined rather easily. The merit of this approach is that a projective transformation is used to convert the pair of input quadrics into forms that are simpler than what are obtained by Levin's method using an affine transformation. But the authors of [24] seem to be unaware of the classical results by Bromwich [7] on classifying the QSIC in complex projective space using the Segre characteristics and the standard technique for simultaneous block diagonalization of two real symmetric matrices [36]. As a consequence, the link between the algebraic structure of the eigenspaces and the type of singularity or reducibility of the intersection curve is not discussed fully in [24]. Furthermore, the procedures presented there for classifying and computing the intersection curve are not thoroughly analyzed; for instance, the case of two quadrics intersecting in a line and a space cubic curve is not accounted for, and a QSIC having two singular points is listed in one of the generic cases and parameterized using a square-root function, although such a curve is reducible and thus comprises a collection of rational curves.

Farouki et al. [12] present another algebraic method that uses rational arithmetic to compute degenerate QSICs. Although degenerate QSICs occupy only a lower-dimension manifold in the configuration space of all QSICs, they occur frequently in practice and allow rational parameterizations. The *Segre characteristic* is used in [12] to classify a degenerate QSIC topologically and detect its degeneracy and reducibility. Given two quadrics $\mathcal{S}_0 : \mathbf{XAX}^{\mathrm{T}} = 0$ and $\mathcal{S}_1 : \mathbf{XBX}^{\mathrm{T}} = 0$, the Segre characteristic is defined by the invariant factors of the quadratic form $\lambda\mathbf{A} + \mu\mathbf{B}$, [7,35, pp. 267–273]. The degeneracy of the QSIC is detected by testing whether or not the discriminant of the characteristic equation $\det(\lambda\mathbf{A} + \mu\mathbf{B}) = 0$ is zero. When the QSIC is found to be degenerate, a quartic *projection cone* is derived for the QSIC. The reducibility of the QSIC is then determined by factoring the quartic projection cone.

The method in [3] uses a projection between points of a plane curve and points of a space curve defined by the transversal intersection of two algebraic surfaces. The mapping is a parallel projection, with the projection direction chosen in a randomized manner to avoid a possible degenerate many-to-one correspondence. When applied to intersecting two quadrics, this method leads to a plane quartic curve. Note that the projections used in both [12] and [3] have the center of projection not on the QSIC.

The representation of the intersection curve of two algebraic surfaces by an algebraic plane curve plus a birational mapping has also been studied in [13]. A family of projections from the points on a line are used to uniquely determine the intersection curve. A parameterization and topological classification of the intersection curve can be achieved by parameterizing and classifying the corresponding plane algebraic curve. This method also needs to analyse a plane quartic curve when the two surfaces are quadrics.

Wilf and Manor [40] extend Levin's method to classify a general QSIC as well as produce its geometric description. To classify a QSIC, the roots of the characteristic equation are solved for numerically, and then the Segre characteristic is found and used to guide the parameterization of the QSIC utilizing Levin's method. Thus their method is a hybrid approach: Levin's method is used for parameterization and the Segre characteristic is used for classification. This method does not compute the number of connected components of a nonsingular QSIC in real projective space, since this information is not provided by the Segre characteristic.

A geometric approach to computing the QSIC is taken in [15,16,21,23,34]. In these methods quadrics of different types are represented in different forms, and different routines are invoked for intersecting different combinations of quadrics. The main advantage of the geometric approach is that geometric insights can help determine the configuration of the intersection curve, and a customized routine can be devised to compute the intersection curve with high accuracy in each particular case. In general, geometric methods are more robust numerically than algebraic methods. But, at present, the geometric approach works well only for *natural quadrics*—planes, spheres, circular cones, and right circular cylinders — which are a special class of quadrics that are frequently used in mechanical CAD.

2. Preliminaries

In this section we shall review some results that will be used later in the paper. In particular, we shall discuss rational quadratic parameterizations of a quadric surface, base points, classification of cubic plane curves and QSICs.

2.1. Quadratic parameterizations

Let $\mathcal{S}$ be a quadric surface represented by $\mathbf{XAX}^{\mathrm{T}} = 0$. $\mathcal{S}$ is nondegenerate if $\mathrm{rank}(\mathbf{A}) = 4$, and is properly degenerate if $\mathrm{rank}(\mathbf{A}) = 3$. Properly degenerate quadrics have one singular point in $\mathcal{PR}^3$, the 3D real projective space, and they consist of all the quadric cones and quadric cylinders in E^3, the 3D real Euclidean space. When $\mathrm{rank}(\mathbf{A}) = 2$ or 1, $\mathcal{S}$ consists of two distinct planes or a double plane. $\mathcal{S}$ is *irreducible* if the quadratic form $\mathbf{XAX}^{\mathrm{T}}$ is irreducible, i.e., $\mathbf{XAX}^{\mathrm{T}}$ cannot be factored over the complex field. A quadric is irreducible if and only if it is nondegenerate or properly degenerate.

Any irreducible quadric can be parameterized as a *faithful* rational quadratic surface [32]. A rational parameterization is *faithful* if there is a one-to-one correspondence between points on the surface and points in the parameter domain, except possibly on a finite number of curves on the surface.

The following procedure for deriving a rational quadratic parameterization of an irreducible quadric is standard [1]. Choose a finite regular point $\mathbf{X}_0 = (x_0, y_0, z_0, w_0)$ on $\mathcal{S}$. Let $\mathbf{NX}^{\mathrm{T}} = 0$ be the plane at infinity, where $\mathbf{N} = (0, 0, 0, 1)$. Let $(r, s, t) = (x, y, z)$ be a homogeneous coordinate system for the plane at infinity, where $(x, y, z, 0)$ are the coordinates of the points on $\mathbf{NX}^{\mathrm{T}} = 0$. Then $\mathbf{T} = (r, s, t, 0)$ is a generic point on $\mathbf{NX}^{\mathrm{T}} = 0$. Represent the line passing through $\mathbf{X}_0$ and $\mathbf{T}$ as $\mathbf{X_T}(u, v) = u\mathbf{X}_0 + v\mathbf{T}$. Then $\mathbf{X_T}(u, v)$ intersects the quadric $\mathcal{S}$ in two points, one of which is $\mathbf{X}_0$. Substituting $\mathbf{X_T}(u, v)$ into $\mathbf{XAX}^{\mathrm{T}} = 0$ and solving the resulting equation in (u, v), we obtain $(u, v) = (1, 0)$, which stands for the intersection at $\mathbf{X}_0$, and $(u, v) = (\mathbf{TAT}^{\mathrm{T}}, -2\mathbf{X}_0\mathbf{AT}^{\mathrm{T}})$, which represents the other intersection, given by

$$\mathbf{P}(r, s, t) = (\mathbf{TAT}^{\mathrm{T}})\mathbf{X}_0 - 2(\mathbf{X}_0\mathbf{AT}^{\mathrm{T}})\mathbf{T}. \tag{2}$$

This equation provides a rational quadratic parameterization of $\mathcal{S}$. Since $\mathbf{P}(r, s, t)$ is obtained by a projection through $\mathbf{X}_0$, the point $\mathbf{X}_0$ is called the center of projection (COP) of $\mathbf{P}(r, s, t)$.

For a point $\mathbf{X}$ on $\mathcal{S}$, the inversion formula to $\mathbf{P}(r, s, t)$ in (2) is

$$\mathbf{T} = w_0\mathbf{X} - w_0\mathbf{X}_0. \tag{3}$$

This formula is a rational linear mapping from $\mathcal{S}$ to the plane at infinity. Thus $\mathbf{P}(r, s, t)$ is a birational mapping between points in the parameter domain and points on the quadric surface $\mathcal{S}$.

2.2. Base points

A *base point* of a rational surface $\mathbf{P}(r, s, t)$ is a parameter point $(r_0, s_0, t_0) \neq 0$ such that $\mathbf{P}(r_0, s_0, t_0) = 0$. If $\mathbf{P}(r, s, t) = (x(r, s, t), y(r, s, t), z(r, s, t), w(r, s, t))$ is a faithful rational quadratic parameterization of an irreducible quadric $\mathcal{S}$: $\mathbf{XAX}^{\mathrm{T}} = 0$, then $\mathbf{P}(r,s,t)$ must have two base points [32]. The two base points can be distinct real points (for hyperbolic paraboloids and their projective equivalents), a double real point (for quadric cones and cylinders), or complex conjugate points (for ellipsoids and their projective equivalents). The *base line* of $\mathbf{P}(r,s,t)$ is the line determined by the two base points of $\mathbf{P}(r,s,t)$. In the case of a double base point, the base line is the unique tangent shared by the four conics $x(r,s,t) = 0$, $y(r,s,t) = 0$, $z(r,s,t) = 0$, and $w(r,s,t) = 0$ at the double base point [39].

Suppose the parameter plane is chosen to be the plane at infinity as in Section 2.1 and the COP of $\mathbf{P}(r,s,t)$ is $\mathbf{X}_0$. Then the base line equation of $\mathbf{P}(r,s,t)$ is $\mathbf{X}_0\mathbf{AT}^{\mathrm{T}} = 0$, where $\mathbf{T} = (r,s,t,0)$; that is, the base line is the intersection between the plane at infinity and the tangent plane of $\mathbf{XAX}^{\mathrm{T}} = 0$ at $\mathbf{X}_0$. Moreover, the base points are the two intersection points of the base line $\mathbf{X}_0\mathbf{AT}^{\mathrm{T}} = 0$ and the conic $\mathbf{TAT}^{\mathrm{T}} = 0$. Treating the base points as points $\mathbf{T}'_0$, and $\mathbf{T}'_1$ in 3D on the plane at infinity, the generating lines of $\mathcal{S}$ passing through $\mathbf{X}_0$ are $\mathbf{X}_0 + v\mathbf{T}'_0$ and $\mathbf{X}_0 + v\mathbf{T}'_1$.

Theorem 1. *Let $\mathbf{P}(r,s,t)$ be a faithful rational quadratic parameterization of a quadric $\mathcal{S}$. An algebraic plane curve $K(r,s,t) = 0$ of degree k is mapped by $\mathbf{P}(r,s,t)$ into a space curve on $\mathcal{S}$ of degree $2k - p$ iff $K(r,s,t) = 0$ passes p times through the base points of $\mathbf{P}(r,s,t)$.*

Proof. Let $\mathcal{C}$ be the image on $\mathcal{S}$ of $K(r,s,t) = 0$ under $\mathbf{P}(r,s,t)$. Then the degree of $\mathcal{C}$ is the number of intersections between $\mathcal{C}$ and a generic plane $\mathbf{BX}^{\mathrm{T}} = 0$. Clearly, the intersections of $\mathcal{C}$ and $\mathbf{BX}^{\mathrm{T}} = 0$ are in one-to-one correspondence with those intersections in the parameter plane between $K(r,s,t) = 0$ and the conic $\mathbf{BP}^{\mathrm{T}}(r,s,t) = 0$ that are not at the base points of $\mathbf{P}(r,s,t)$, since a base point is not mapped by $\mathbf{P}(r,s,t)$ into any well-defined point.

By Bezout's theorem, there are $2k$ intersections between $K(r,s,t) = 0$ and the conic $\mathbf{BP}^{\mathrm{T}}(r,s,t) = 0$. Moreover $K(r,s,t) = 0$ passes through the base points p times if and only if exactly p of these $2k$ intersections are at the base points for all $\mathbf{B} \neq 0$, or equivalently, the number of intersections between $K(r,s,t) = 0$ and $\mathbf{BP}(r,s,t)^{\mathrm{T}} = 0$ that are not at the base points is $2k - p$. $\square$

By Theorem 1, a line in the (r,s,t) plane is, in general, mapped by $\mathbf{P}(r,s,t)$ into a conic on $\mathcal{S}$. But if the line passes through exactly one base point, it will be mapped by $\mathbf{P}(r,s,t)$ into a straight line on $\mathcal{S}$. The base line, which passes through two base points, is mapped into a point on $\mathcal{S}$, which is, in fact, the COP of $\mathbf{P}(r,s,t)$. Similarly, a proper conic in the (r,s,t) plane could be mapped into a quartic curve, a cubic curve, or a conic on $\mathcal{S}$, depending on how many times it passes through the base points of $\mathbf{P}(r,s,t)$. This analysis will help us compute the irreducible components of a QSIC by mapping the corresponding components of the associated plane cubic curve by $\mathbf{P}(r,s,t)$.

2.3. Classification of plane cubic curves

Plane cubic curves have been well studied in algebraic geometry [33,38] and in CAGD [26], As our algorithm is based on the analysis of plane cubic curves, we briefly review the classification of plane cubic curves. We shall assume that the coefficients of a cubic curve are real numbers. Note that there are always real points on such a plane cubic curve.

A reducible plane cubic curve consists of either three lines or a line and a conic. In the former case, two of the three lines might form a conjugate pair. In the latter case, the conic might be imaginary, i.e., have no real points. There exist simple techniques to compute rational parameterizations for the real linear and conic components of a reducible cubic curve [1].

An irreducible plane cubic curve can be transformed projectively into the following form in homogeneous coordinates (r, s, t) [5],

$$ts^2 = ar^3 + br^2t + crt^2 + dt^3. \tag{4}$$

Five main species of cubic curves can be distinguished according to the zeroes of the cubic polynomial on the right-hand side: (a) a singular cubic curve with a crunode; (b) a singular cubic curve with a cusp; (c) a singular cubic curve with an acnode; (d) a nonsingular cubic curve with two connected components in $\mathcal{PR}^2$; (e) a nonsingular cubic curve with one component in $\mathcal{PR}^2$. These five cases are illustrated in Fig. 1(a)–(e).

An irreducible plane cubic curve can be singular or nonsingular. A singular cubic curve has exactly one double point, which can be either a cusp, a crunode, or an acnode. A singular cubic curve has a rational cubic parameterization, which can be obtained from a pencil of lines centered at the double point. A nonsingular plane cubic can have one or two connected real components in $\mathcal{PR}^2$. Of particular interest to us are the nonsingular cubic curves with two connected components in $\mathcal{PR}^2$. One of these components is called the *infinite component*, the other the *oval*. The infinite component is characterized by the fact that it is intersected by every line in $\mathcal{PR}^2$; an intersection point may be at infinity in an affine realization of $\mathcal{PR}^2$. A nonsingular plane cubic curve does not have a rational parameterization, though it can be parameterized with square roots [2] or elliptic functions [25]. The reader is referred to [29] for more information about the classification of plane cubic curves and to [26] for a discussion of singular cubic curves.

2.4. Classification of QSICs

There are many configurations of QSICs resulting from the intersection of different quadrics with different relative positions and orientations. In general, a QSIC is a space quartic curve of the first species; an arbitrary plane not containing any component of the QSIC intersects it in exactly four points with multiplicities, complex intersections, and intersections at infinity counted properly.

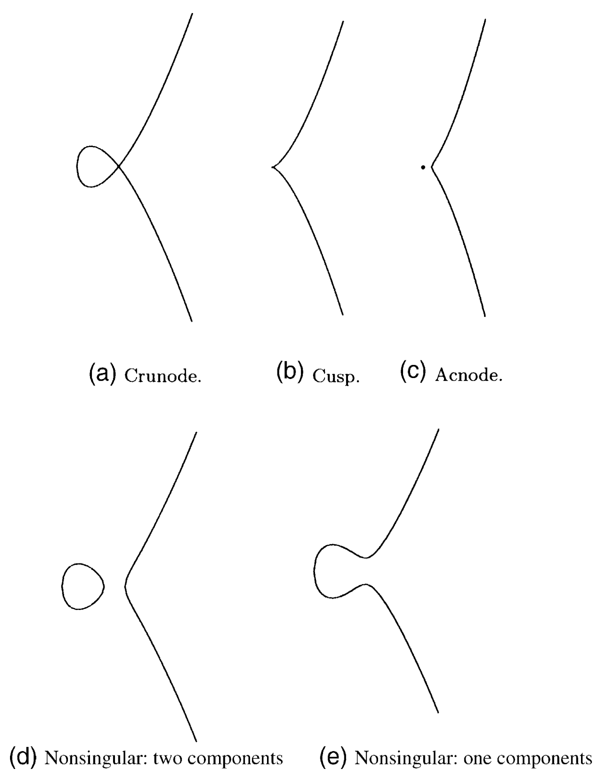

Fig. 1. Five species of irreducible plane cubic curves.

The composition of a reducible QSIC can be in one of the following configurations [4,12]: (1) *four lines*: the four lines are either four real lines, or two real lines plus a pair of complex conjugate lines, or two pairs of complex conjugate lines. No more than two of these four lines can be mutually skew; that is, the four lines must form two degenerate conics; (2) *two lines and one conic*: the two lines are either two real lines or a pair of complex conjugate lines, and the conic can be real or imaginary; (3) *one line and one cubic space curve*: the line and the cubic space curve must both be real, and the line must either be tangential to the cubic or intersect the cubic in two distinct points; (4) *two conics*: either of the two conics can be real or imaginary.

An irreducible QSIC can be singular. A singular QSIC has exactly one singular point, and has a rational quartic parameterization. This singular point can be a node or a cusp on the QSIC; it can also be an isolated real point, similar to an acnode on a singular plane cubic curve. A nonsingular QSIC has no singular point and does not admit a rational parameterization, but it has a parameterization with square roots. We will prove later that a nonsingular QSIC in $\mathcal{PR}^3$ has exactly one or two connected components, as commonly believed. These properties of QSICs are reminiscent of those of plane cubic curves.

3. Computing the projection of a QSIC

Given two quadrics, we shall discuss in this section: (1) how to derive the cubic plane curve corresponding to the intersection of the two quadrics; and (2) how to process this cubic curve. This discussion is necessary before we can present the intersection algorithm in Section 4.

3.1. Deriving the plane cubic curve

Let $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$ be two distinct quadrics. Assume that at least one of $F(\mathbf{X}) = 0$ or $G(\mathbf{X}) = 0$ is irreducible; otherwise the problem of computing the QSIC can be reduced to finding intersections between two pairs of planes. Without loss of generality we assume that $F(\mathbf{X}) = 0$ is irreducible, by swapping $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$ if necessary. Let $\mathbf{X}_0$ be a regular point on $F(\mathbf{X}) = 0$, and let $\mathbf{P}(r,s,t)$ be a faithful rational quadratic parameterization of $F(\mathbf{X}) = 0$ with its COP at $\mathbf{X}_0$. Let the QSIC refer to the intersection curve of $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$. Then the QSIC corresponds to the quartic curve

$$\bar{G}(r,s,t) \equiv G(\mathbf{P}(r,s,t)) = 0 \tag{5}$$

in the (r,s,t) plane. In general, there is a one-to-one correspondence between points on the QSIC and points on $\bar{G}(r,s,t) = 0$. The exceptional case is when either of the two generating lines of $F(\mathbf{X}) = 0$ passing through $\mathbf{X}_0$ is part of the QSIC, since all points on such a line correspond to a base point under $\mathbf{P}(r,s,t)$. The QSIC is mapped by the inverse of $\mathbf{P}(r,s,t)$ into the components of the quartic curve $\bar{G}(r,s,t) = 0$. We shall now examine conditions under which the quartic curve $\bar{G}(r,s,t) = 0$ can be simplified.

Theorem 2. *Let $L(r,s,t) = 0$ be a linear equation of the base line of $\mathbf{P}(r,s,t)$. Then $L(r,s,t)$ is a k-fold factor of $\bar{G}(r,s,t)$ if and only if $\mathbf{X}_0$ is a k-fold point of the QSIC.*

Proof. The following is proved in [39]. Given any faithful parameterization $\mathbf{P}(r,s,t)$ of an irreducible quadric $\mathcal{S}$, there is a plane $\mathcal{P}$ not passing through $\mathbf{X}_0$ such that $\mathbf{P}(r,s,t)$ is a perspective projection from $\mathcal{P}$ onto $\mathcal{S}$, and the COP of this projection is on $\mathcal{S}$. This perspective projection and the parameter plane associated with $\mathbf{P}(r,s,t)$ are used in the following argument.

Let $\mathbf{X}_0$ be a k-fold point of the QSIC, $0 \leqslant k \leqslant 4$, and let $L(r,s,t)$ be an ℓ-fold factor of $\bar{G}(r,s,t)$. Then $\bar{G}(r,s,t) = L^{\ell}(r,s,t)K(r,s,t)$, where $K(r,s,t)$ is a polynomial of degree $4-\ell$.

Let $\mathcal{L}$ be a generic line in the parameter plane. Then the plane determined by $\mathcal{L}$ and $\mathbf{X}_0$ intersects the QSIC at $4-k$ points other than $\mathbf{X}_0$. These $4-k$ intersections are in a one-to-one correspondence with the intersections of $\mathcal{L}$ and $\bar{G}(r,s,t) = 0$ that are not on the base line $L(r,s,t) = 0$, i.e., the intersections between $\mathcal{L}$ and $K(r,s,t) = 0$. Since the number of intersections of a generic line with an algebraic plane curve is equal to the degree of the curve, we have $4-k = 4-\ell$; that is, $k = \ell$. $\square$

Theorem 2 implies that if the parameterization $\mathbf{P}(r,s,t)$ of $F(\mathbf{X})=0$ is chosen appropriately, i.e., with its COP being a regular point of the QSIC, the quartic curve $\bar{G}(r,s,t)=0$ always contains the base line as a linear component. The points on the remaining cubic component $K(r,s,t)=0$ are essentially in a one-to-one correspondence with all the points of the QSIC, since the base line is always mapped into the COP of $\mathbf{P}(r,s,t)$. As the base line equation $L(r,s,t)=0$ is easy to obtain (see Section 2), the remaining component $K(r,s,t)=0$ can be computed by polynomial division.

Let $\mathbf{F}'(\mathbf{X})=(\partial F/\partial x, \partial F/\partial y, \partial F/\partial z, \partial F/\partial w)$ and $\mathbf{G}'(\mathbf{X})=(\partial G/\partial x, \partial G/\partial y, \partial G/\partial z, \partial G/\partial w)$. The point $\mathbf{X}_0$ is a regular point of the QSIC if and only if $F(\mathbf{X}_0)=G(\mathbf{X}_0)=0$ and $F'(\mathbf{X}_0)$ and $G'(\mathbf{X}_0)$ are not collinear. These constraints constitute easy-to-test conditions for determining whether or not $\mathbf{X}_0$ is a regular point of the QSIC.

When the COP $\mathbf{X}_0$ is a singular point of the QSIC, by Theorem 2, $L(r,s,t)$ is a multiple factor of $\bar{G}(r,s,t)$ and the remaining factor $K(r,s,t)$ is at most quadratic. In this case, all irreducible components of $K(r,s,t)=0$ are mapped to irreducible components of the QSIC.

The point $\mathbf{X}_0$ chosen to be the COP may happen to be a singular point or may be a singular point by design. For instance, if the QSIC contains a singular point, then this singular point could be found by some method based on Levin's algorithm and could be used as a COP in our algorithm. Since a singular COP is a special case, we require only that an arbitrary point on the QSIC be used as a COP, without requiring it to be at the singular point even when the QSIC is singular. When a regular point of a QSIC is used as the COP, detecting the singularity of the QSIC is reduced to detecting the singularity of a plane cubic curve. Hence our treatment is simpler than computing the singularity of a QSIC directly by Levin's method. In the following we will mainly discuss the case where the COP of $\mathbf{P}(r,s,t)$ is a regular point on the QSIC.

Let the COP of $\mathbf{P}(r,s,t)$ be a regular point on the QSIC. By Theorem 2, $\bar{G}(r,s,t)=L(r,s,t)K(r,s,t)$, where $K(r,s,t)$ is a cubic factor of $\bar{G}(r,s,t)$ not containing $L(r,s,t)$.

Theorem 3. *The cubic curve $K(r,s,t)=0$ passes at least once through each of the two base points of $\mathbf{P}(r,s,t)$. When $\mathbf{P}(r,s,t)$ has a double base point, $K(r,s,t)=0$ passes through the base point at least twice.*

Proof. There are two cases to consider: (a) $\mathbf{P}(r,s,t)$ has two distinct base points; (b) $\mathbf{P}(r,s,t)$ has a double base point.

Case (a): Let $\mathbf{b}_0$ be a base point of $\mathbf{P}(r,s,t)$. Since all four component functions of $\mathbf{P}(r,s,t)$ vanish at $\mathbf{b}_0$ and $G(\mathbf{X})$ is quadratic, $\bar{G}(r,s,t)\equiv G(\mathbf{P}(r,s,t))=0$ is singular at $\mathbf{b}_0$. Since $L(r,s,t)=0$ is regular at $\mathbf{b}_0$ and $\bar{G}(r,s,t)=L(r,s,t)K(r,s,t)$, $K(r,s,t)$ must vanish at $\mathbf{b}_0$, for otherwise $\bar{G}(r,s,t)=0$ would not be singular at $\mathbf{b}_0$. Hence $K(r,s,t)=0$ passes through $\mathbf{b}_0$.

Case (b): If $K(r,s,t)=0$ is mapped by $\mathbf{P}(r,s,t)$ into a quartic QSIC on $F(\mathbf{X})=0$, then, by Theorem 1, $K(r,s,t)=0$ must pass through the double base point twice. If

$K(r,s,t) = 0$ is mapped into a lower-degree component of the QSIC, then, by Theorem 1, it must pass through the base point more than twice. $\square$

3.2. Processing the cubic curve

Again suppose that the COP of $\mathbf{P}(r,s,t)$ is a regular point on the QSIC. By Theorem 2, the base line component $L(r,s,t) = 0$ can be removed from $\bar{G}(r,s,t) = 0$, and the remaining cubic component $K(r,s,t) = 0$ does not contain $L(r,s,t) = 0$. Now consider detecting the reducibility and singularity of the cubic $K(r,s,t) = 0$ in order to detect the reducibility and singularity of the QSIC. Note that if $K(r,s,t) = 0$ is reducible, the QSIC is also reducible. It is also useful to detect the singularity of an irreducible QSIC, since in this case the QSIC possesses a rational quartic parameterization [12].

In our algorithm the reducibility of the cubic curve $K(r,s,t) = 0$ is detected first. If it is irreducible, we further detect its singularity as follows. By definition, a singular point of $K(r,s,t) = 0$ is a common intersection of the three conics $\partial K(r,s,t)/\partial r = 0$, $\partial K(r,s,t)/\partial s = 0$, and $\partial K(r,s,t)/\partial t = 0$. Denote these three conics by $\mathcal{C}_0 : \mathbf{r}\mathbf{H}_0\mathbf{r}^{\mathrm{T}} = 0$, $\mathcal{C}_1 : \mathbf{r}\mathbf{H}_1\mathbf{r}^{\mathrm{T}} = 0$, and $\mathcal{C}_2 : \mathbf{r}\mathbf{H}_2\mathbf{r}^{\mathrm{T}} = 0$, where $\mathbf{r} = (r,s,t)$ and the $\mathbf{H}_i$, $i = 0,1,2$, are real 3×3 symmetric matrices. Because $K(r,s,t) = 0$ is irreducible, the $\mathcal{C}_i$, $i = 0,1,2$, cannot be the same. Without loss of generality, assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct conics. By solving for a real root λ_1 of the cubic equation $|\lambda\mathbf{H}_1 + \mathbf{H}_2| = 0$, we obtain a reducible conic $\mathbf{r}(\lambda_1\mathbf{H}_1 + \mathbf{H}_2)\mathbf{r}^{\mathrm{T}} = 0$, whose two constituent lines are denoted by $\mathcal{L}_1$ and $\mathcal{L}_2$. Then the four intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$ are the intersections of $\mathcal{L}_1$ and $\mathcal{L}_2$ with $\mathcal{C}_1$ (or $\mathcal{C}_2$). Among the four intersections, only those lying on $\mathcal{C}_0 : \mathbf{r}\mathbf{H}_0\mathbf{r}^{\mathrm{T}} = 0$ are singular points of $K(r,s,t) = 0$. Since a singular point of an irreducible $K(r,s,t) = 0$ must be real, only real intersections need to be considered. We may also assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are real lines; when they are complex conjugate, we just test whether or not their common real point is a singular point of the cubic $K(r,s,t) = 0$. Note that at most one singular point is expected, since $K(r,s,t) = 0$ is irreducible.

Another method which assumes only rational arithmetic for computing the singularity of general algebraic plane curves is proposed in [30]. Yet another possible method is to use resultants to determine if $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$ have a common intersection. Since we consider only a floating point implementation of our algorithm, the method described above suffices.

When $K(r,s,t) = 0$ is irreducible and singular, the type of its double point $\mathbf{r}_0$ is determined by the Hessian $\mathbf{H(r)}$ of $K(r,s,t) = 0$ at $\mathbf{r}_0$ where

$$\mathbf{H}(\mathbf{r}) = \begin{bmatrix} \frac{\partial K(\mathbf{r})}{\partial^2 r} & \frac{\partial K(\mathbf{r})}{\partial r \partial s} & \frac{\partial K(\mathbf{r})}{\partial r \partial t} \\ \frac{\partial K(\mathbf{r})}{\partial s \partial r} & \frac{\partial K(\mathbf{r})}{\partial^2 s} & \frac{\partial K(\mathbf{r})}{\partial s \partial t} \\ \frac{\partial K(\mathbf{r})}{\partial t \partial r} & \frac{\partial K(\mathbf{r})}{\partial t \partial s} & \frac{\partial K(\mathbf{r})}{\partial^2 t} \end{bmatrix}$$

with $\mathbf{r} = (r,s,t)$. Since $K(r,s,t) = 0$ is an irreducible cubic that is singular at $\mathbf{r}_0$, the conic $\mathcal{C}' : \mathbf{r}\mathbf{H}(\mathbf{r}_0)\mathbf{r}^{\mathrm{T}} = 0$ is a reducible conic. The point $\mathbf{r}_0$ is a cusp, an acnode, or a crunode of $K(r,s,t) = 0$ if the conic $\mathbf{r}\mathbf{H}(\mathbf{r}_0)\mathbf{r}^{\mathrm{T}} = 0$ consists of a double real line, a pair of conjugate lines, or a pair of distinct lines, respectively. We will, however, use a simpler expression in the next section to classify the type of a singular point of $K(r, s, t) = 0$.

4. Computing the QSIC

In this section we shall first give an outline of our method, followed by detailed discussions on three main issues: (1) parameterization; (2) classification; and (3) detection of lines contained in the QSIC.

4.1. Outline of the algorithm

Let $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$ be two distinct quadrics. Assume that $F(\mathbf{X}) = 0$ is irreducible. Our algorithm consists of the following steps.

1. Compute a real point $\mathbf{X}_0$ of the intersection of $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$ such that $\mathbf{X}_0$ is a regular point on $F(\mathbf{X}) = 0$. If such a point does not exist, the QSIC is either empty or consists of only the singular point of $F(\mathbf{X}) = 0$; so quit.
2. Use $\mathbf{X}_0$ as the COP to derive a faithful rational quadratic parameterization $\mathbf{P}(r, s, t)$ of $F(\mathbf{X}) = 0$. Substitute $\mathbf{P}(r, s, t)$ into $G(\mathbf{X}) = 0$ to get a quartic plane curve $\bar{G}(r, s, t) \equiv G(\mathbf{P}(r, s, t)) = 0$. Remove a linear component—the base line of $\mathbf{P}(r, s, t)$—from $\bar{G}(r, s, t) = 0$ to get the remaining component, denoted by $K(r, s, t) = 0$, such that $K(r, s, t) = 0$ does not contain the base line.
3. Factor $K(r, s, t) = 0$ into irreducible components. When $K(r, s, t) = 0$ is irreducible and $K(r, s, t)$ is cubic, use the method in Section 3.2 to detect whether or not $K(r, s, t) = 0$ is singular.
4. Parameterize each irreducible component of $K(r, s, t) = 0$. Map the parametric equations of these components by $\mathbf{P}(r, s, t)$ into parameterizations of the corresponding components of the QSIC. Eliminate a common factor of the resulting parameterized QSIC components if their corresponding components in $K(r, s, t) = 0$ pass through the base points of $\mathbf{P}(r, s, t)$.
5. Determine if any of the two generating lines of $F(\mathbf{X}) = 0$ passing through $\mathbf{X}_0$ is contained in the QSIC.

Step 1 requires us to find a point on the QSIC. This point is computed in our algorithm using Levin's method, which has proven to be numerically robust [31]. Specifically, we first use Levin's method to find a parameterization of the QSIC in the form $\mathbf{Q}(u, v) = \mathbf{S}(u, v) \pm \mathbf{T}(u, v)\sqrt{D(u, v)}$. Then, by solving the quartic equation $D(u, v) = 0$, we determine an interval of the homogeneous parameter (u, v) over which $D(u, v) \geqslant 0$; an arbitrary parameter (u_0, v_0) in this interval is used to yield a real point $\mathbf{Q}(u_0, v_0)$ on the QSIC. If $D(u, v) = 0$ has no real roots and $D(u, v) > 0$ for all parameters (u, v), then any parameter (u, v) can be used to yield a real point $\mathbf{Q}(u_0, v_0)$. But if $D(u, v) = 0$ has no real roots and $D(u, v) < 0$ for all parameters (u, v), then in this case the input quadrics $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$ have no real intersection points [18]. As pointed out in [18, p. 558], [19, p. 76], another case in which the two quadrics can be detected to have no real intersection points is when an *invalid surface* (i.e., an imaginary quadric given by a definite matrix) is found during the search for a parameterization surface (i.e., a ruled quadric) in the pencil of $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$. Hence, by testing if either of these two cases occurs, Levin's method also reports correctly when the two quadrics do not have any real intersection point. The following fact, implied but not proved in [18], is useful in understanding the condition for detecting the lack of real intersection points between two real quadrics: two real quadrics have no real intersection points if and only if the pencil of their coefficient matrices contains a definite matrix. See [9] for a proof of this result.

Our algorithm works correctly even when all the points of the QSIC are singular points of the QSIC, since it just requires that the COP of $F(\mathbf{X}) = 0$ be a regular point of the quadric surface $F(\mathbf{X}) = 0$; the COP does not have to be a regular point of the QSIC. The simple example below demonstrates this condition as well as the main steps of the algorithm. Several other more general examples will be presented in Section 5.

Consider two quadrics $F(\mathbf{X}) = \mathbf{XAX}^{\mathrm{T}} \equiv x^2 + y^2 + z^2 - w^2 = 0$ and $G(\mathbf{X}) = \mathbf{XBX}^{\mathrm{T}} = 4x^2 + 4y^4 + z^2 - w^2 = 0$. Clearly, $G(\mathbf{X}) = 0$ is an ellipsoid contained in the sphere $F(\mathbf{X}) = 0$, and there are two real touching points between the two surfaces. Therefore there are only two real points $\mathbf{V}_0 = (0, 0, 1, 1)$ and $\mathbf{V}_1 = (0, 0, -1, 1)$ on the QSIC of $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$, and these two points are singular points of the QSIC. Now let us see how these two points are computed by our algorithm. In step one, Levin's method is used to find the point $\mathbf{V}_0$ on the QSIC, and it can be verified that $\mathbf{V}_0$ is a double point of the QSIC but a regular point of the quadric $F(\mathbf{X}) = 0$. In step two, using $\mathbf{V}_0$ as the COP of $F(\mathbf{X}) = 0$, we find a parameterization $\mathbf{P}(r, s, t)$ of $F(\mathbf{X}) = 0$:

$$\mathbf{P}(r, s, t) = (-2rt, -2st, r^2 + s^2 - t^2, r^2 + s^2 + t^2).$$

Substituting $\mathbf{P}(r, s, t)$ in $G(\mathbf{X}) = 0$, we obtain the plane quartic curve $\bar{G}(r, s, t) \equiv 12(r^2 + s^2)t^2 = 0$. Since $\mathbf{V}_0$, the COP of $F(\mathbf{X}) = 0$, is a double singular point of the QSIC, by Theorem 2, we may remove the base line factor $\mathbf{V}_0\mathbf{A}(r, s, t, 0)^{\mathrm{T}} = t$ twice from $\bar{G}(r, s, t) = 0$ to get the conic curve $K(r, s, t) = 12(r^2 + s^2) = 0$. In step three, the curve $K(r, s, t) = 0$ can be factored into two conjugate complex lines $r + \mathrm{i}s = 0$ and $r - \mathrm{i}s = 0$, which intersect at the real point $(0, 0, 1)$; these two lines can be mapped by $\mathbf{P}(r, s, t)$ onto the quadric $F(\mathbf{X}) = 0$ to get the two degree-two components of the QSIC, and the only real point $(0, 0, 1)$ on $K(r, s, t) = 0$ is mapped by $\mathbf{P}(r, s, t)$ into the real intersection point $\mathbf{V}_1 = (0, 0, -1, 1)$ of the QSIC. Hence we have obtained all the real intersection points between $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$.

The computation in step 2 of $\mathbf{P}(r, s, t)$ and its base points and base line has been described in Section 2. It is straightforward to compute $\bar{G}(r, s, t)$ and remove the base line factor. It is also straightforward to factor $K(r, s, t)$ as a cubic bivariate polynomial. The basic idea is to determine a linear polynomial $C(r, s, t)$ and a quadratic polynomial $D(r, s, t)$ such that $C(r, s, t)D(r, s, t) = K(r, s, t)$. In the remainder of this section we shall discuss steps 4 and 5, and the identification of connected components of a nonsingular QSIC.

4.2. Parameterizing the QSIC

We suppose in the following that $K(r,s,t)=0$ is a plane cubic curve, i.e., the COP of $\mathbf{P}(r,s,t)$ is a regular point of the QSIC; for otherwise $K(r,s,t)=0$ is a conic or a line, which can easily be classified and parameterized as a rational curve.

Parameterization is done by first computing a parameterization of the cubic curve $K(r,s,t)=0$, and then mapping it by $\mathbf{P}(r,s,t)$ into a parameterization of the QSIC. Since, by Theorem 3, $K(r,s,t)=0$ passes through the base points of $\mathbf{P}(r,s,t)$, all coordinate components of this parameterization of the QSIC share a common factor. Such a parameterization is problematic when used to compute points on the QSIC. We shall only discuss parameterization in two cases: (1) $K(r,s,t)=0$ is singular and irreducible; and (2) $K(r,s,t)=0$ is nonsingular. We skip the case where $K(r,s,t)=0$ is reducible, since in this case all its conic or linear components can readily be parameterized with rational functions, and the procedure for elimination of a common factor is similar to the previous cases.

4.2.1. Parameterization of a singular QSIC

Suppose $K(r,s,t)=0$ is a singular but irreducible plane cubic curve with the double point $\mathbf{r}_0=(r_0,s_0,t_0)$. Without loss of generality, we may assume $t_0\neq 0$, so we may set $\mathbf{u}=(u,v,0)$ below; if $t_0=0$, the three components of $\mathbf{r}_0=(r_0,s_0,t_0)$ may be permuted into $\mathbf{r}_0'=(r_0',s_0',t_0')$ with $t_0'\neq 0$, and the same procedure can be followed. The pencil of lines centered at $\mathbf{r}_0$ is

$$\mathbf{w}(\mathbf{u};\mu,\lambda)=\mu\mathbf{r}_0+\lambda\mathbf{u}. \tag{6}$$

Substituting $\mathbf{w}(\mathbf{u};\mu,\lambda)$ into $K(r,s,t)=0$ and using Taylor's expansion for multivariate polynomials, we have

$$K(\mu\mathbf{r}_0+\lambda\mathbf{u})=\mu^3K(\mathbf{r}_0)+\mu^2\lambda\left(\frac{\partial K}{\partial r}u+\frac{\partial K}{\partial s}v\right)+\frac{1}{2!}\mu\lambda^2\sum_{i=0}^{2}\frac{2!}{i!(2-i)!}\times\frac{\partial^2K}{\partial^{2-i}r\partial^is}u^{2-i}v^i+\frac{1}{3!}\lambda^3\sum_{i=0}^{3}\frac{3!}{i!(3-i)!}\frac{\partial^3K}{\partial^{3-i}r\partial^is}u^{3-i}v^i=0,$$

where all the derivatives of $K(r,s,t)$ are evaluated at $\mathbf{r}_0$. Since $\mathbf{r}_0$ is a singular point of $K(r,s,t)=0$, $K(\mathbf{r}_0)=\partial K/\partial r=\partial K/\partial s=0$. Dropping the trivial factor $\lambda^2=0$, we obtain

$$\mu B(u,v)+\lambda C(u,v)=0,$$

where

$$B(u,v)=3\sum_{i=0}^{2}\frac{2!}{i!(2-i)!}\frac{\partial^2K}{\partial^{2-i}r\partial^is}u^{2-i}v^i,$$

$$C(u,v)=\sum_{i=0}^{3}\frac{3!}{i!(3-i)!}\frac{\partial^3K}{\partial^{3-i}r\partial^is}u^{3-i}v^i.$$

Substituting $\mu = C(u, v)$ and $\lambda = -B(u, v)$ in $\mathbf{w}(\mathbf{u}; \mu, \lambda)$, we obtain a rational cubic parameterization of $K(r, s, t) = 0$,

$$\mathbf{r}(u, v) = C(u, v)\mathbf{r}_0 - B(u, v)(u, v, 0).$$

Suppose the parameterization $\mathbf{P}(r, s, t)$ of $F(\mathbf{X}) = 0$ is expressed in the quadratic form

$$\mathbf{P}(r, s, t) = (r, s, t)\mathbf{M}(r, s, t)^{\mathrm{T}},$$

where $\mathbf{M} = [\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3]$ denotes four 3×3 symmetric matrices $\mathbf{M}_i$, $i = 0, 1, 2, 3$, and the above equation is a short-hand notation for

$$\mathbf{P}(r, s, t) = [(r, s, t)\mathbf{M}_0(r, s, t)^{\mathrm{T}}, (r, s, t)\mathbf{M}_1(r, s, t)^{\mathrm{T}},$$
$$(r, s, t)\mathbf{M}_2(r, s, t)^{T}, (r, s, t)\mathbf{M}_3(r, s, t)^{\mathrm{T}}].$$

Mapping $\mathbf{r}(u, u)$ by $\mathbf{P}(r, s, t)$ onto $F(\mathbf{X}) = 0$, we obtain a rational parameterization of the QSIC

$$\tilde{\mathbf{Q}}(u, v) = \mathbf{P}(\mathbf{r}(u, v)) = \mathbf{r}(u, v)\mathbf{M}\mathbf{r}^{\mathrm{T}}(u, v),$$

which is a rational parameterization of degree 6. By Theorem 3, $\mathbf{r}(u, v)$ passes at least once through each of the two base points of $\mathbf{P}(r, s, t)$. Let (u_0, v_0) and (u_1, v_1) be the values of (u, v) such that $\mathbf{r}(u_0, v_0)$ and $\mathbf{r}(u_1, v_1)$ are the two base points of $\mathbf{P}(r, s, t)$. Then $\tilde{\mathbf{Q}}(u_0, v_0) = \tilde{\mathbf{Q}}(u_1, v_1) = 0$. So the factor $(v_0 u - u_0 v)(v_1 u - u_1 v)$ can be removed from all four components of $\tilde{\mathbf{Q}}(u, v)$. Thus the QSIC has the rational quartic parameterization

$$\mathbf{Q}(u, v) = \frac{\mathbf{r}(u, v)\mathbf{M}\mathbf{r}^{\mathrm{T}}(u, v)}{(v_0 u - u_0 v)(v_1 u - u_1 v)}.$$

The factors $(v_0 u - u_0 v)(v_1 u - u_1 v)$ can be found as follows. Let $\mathbf{l}_0(r, s, t)^{\mathrm{T}} = 0$ be the base line equation of $\mathbf{P}(r, s, t)$, where $\mathbf{l}_0$ is a coefficient vector. Then $(v_0 u - u_0 v)$ $(v_1 u - u_1 v)$ is a factor of the cubic polynomial $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v)$. If $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v)$ has three real zeros, $(v_0 u - u_0 v)(v_1 u - u_1 v)$ is determined by the two zeros that give the two base points. If $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v)$ has only one real zero, $(v_0 u - u_0 v)(v_1 u - u_1 v)$ is the remaining quadratic factor of $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v) = 0$, and in this case the two base points are complex conjugates.

If $K(r, s, t) = 0$ passes through the base points more than twice, then it intersects the base line exactly three times at the base points, so all the zeros of $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v) = 0$ correspond to base points. Hence the cubic factor $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v)$ needs to be removed from $\tilde{\mathbf{Q}}(u, v)$. The QSIC then has the rational cubic parameterization

$$\mathbf{Q}(u, v) = \frac{\mathbf{r}(u, v)\mathbf{M}\mathbf{r}^{\mathrm{T}}(u, v)}{\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u, v)}.$$

In this case the QSIC consists of $\mathbf{Q}(u, v)$ and a generating line of $F(\mathbf{X}) = 0$ passing through $\mathbf{X}_0$; the latter can be identified by the method in Section 4.4.

When the QSIC is singular and irreducible, the type of its singular point can easily be determined by the discriminant $\Delta(u, v)$ of the quadratic equation $B(u, v) = 0$: it is a crunode, a cusp, or an acnode if $\Delta(u, u) > 0, = 0$, or < 0, respectively. That is because if $B(u, v) = 0$, the line $\mu\mathbf{r}_0 + \lambda\mathbf{u}$ has at least second order contact with $K(r, s, t) = 0$ at $\mathbf{r}_0$, where $\mathbf{u} = (u, v, 0)$. Therefore the roots of $B(u, v) = 0$ give the directions of the tangents lines of $K(r, s, t) = 0$ at $\mathbf{r}_0$.

4.2.2. Parameterization of a nonsingular QSIC

A nonsingular QSIC will be obtained as the image of a nonsingular plane cubic curve under $\mathbf{P}(r, s, t)$. Such a QSIC cannot be parameterized as a rational curve, but can be parameterized using elliptic functions [25] or with a square root [2]. A parameterization using a square root is computed in our method. Our technique is based on a pencil of lines centered at a point on the cubic curve, but is different from that in [2]; rather than choose the pencil center at infinity as in [2], we choose the center at a special point to ensure that the common factor of the resulting parameterization of the QSIC can be eliminated.

In theory an irreducible plane cubic curve can be transformed into the standard form (4), which will then lead to a simple parameterization of the curve with a square root. One way of deriving the standard form is by first computing an inflection point on the planar cubic curve; an inflection of a planar cubic can be found by solving an equation of degree 9, which is the resultant of the cubic and its Hessian [5]. Identifying inflection points on a curve is an important problem that has been studied in CAGD (see, for example, [20]), but there seems to be no good solution to the problem of finding an inflection point on an irreducible planar cubic curve without solving a degree 9 equation. Another way of obtaining the standard form is by applying a quadratic transformation to the cubic, as described in [25]. This quadratic transformation makes the processing of a parameterization of a planar cubic rather involved, and it does not preserve the projective properties of the curve. Hence we employ in the present paper an approach that needs the solution of only a quartic equation and does not use the involved quadratic transformation. A more detailed comparison of these three different methods for parameterizing a planar cubic curve, in terms of efficiency, accuracy, and simplicity of implementation, is beyond the scope of this paper but would be an interesting topic for further research.

Let $\mathbf{b}_0$ and $\mathbf{b}_1$ be the two base points of $\mathbf{P}(r, s, t)$. By Theorem 3, $\mathbf{b}_0$ and $\mathbf{b}_1$ are two of the three intersection points of $K(r, s, t) = 0$ with the base line $L(r, s, t) = 0$ of $\mathbf{P}(r, s, t)$. Let $\mathbf{r}_0 = (r_0, s_0, t_0)$ be the intersection point of $K(r, s, t) = 0$ and $L(r, s, t) = 0$ which is different from $\mathbf{b}_0$ or $\mathbf{b}_1$. Here we may assume that not all the intersections of $K(r, s, t) = 0$ and $L(r, s, t) = 0$ are at the base points of $\mathbf{P}(r, s, t)$; for otherwise, by Theorems 7 and 8 to be proved in Section 4.4, a generating line of $F(\mathbf{X}) = 0$ passing through $\mathbf{X}_0$ would be included in the QSIC, contradicting our assumption that the QSIC is irreducible.

Again, we may assume that $t_0 \neq 0$ and $\mathbf{u} = (u, v, 0)$, as in Section 4.2.1. So the pencil of lines centered at $\mathbf{r}_0$ is $\mathbf{w}(\mathbf{u}; \mu, \lambda) = \mu\mathbf{r}_0 + \lambda\mathbf{u}$. Similar to the argument in Section 4.2.1, substituting $\mathbf{w}(\mathbf{u}; \mu, \lambda)$ into $K(r, s, t) = 0$, noting that $K(\mathbf{r}_0) = 0$, and dropping the factor λ, we obtain

$$\mu^2\left(\frac{\partial K}{\partial r}u+\frac{\partial K}{\partial s}v\right)+\frac{1}{2!}\mu\lambda\sum_{i=0}^{2}\frac{2!}{i!(2-i)!}\frac{\partial^2 K}{\partial^{2-i}r\partial^i s}u^{2-i}v^i$$
$$+\frac{1}{3!}\lambda^2\sum_{i=0}^{3}\frac{3!}{i!(3-i)!}\frac{\partial^3 K}{\partial^{3-i}r\partial^i s}u^{3-i}v^i=0,$$

or

$$A(u,v)\mu^2+B(u,v)\mu\lambda+C(u,v)\lambda^2=0, \tag{7}$$

where

$$A(u,v)=6\left(\frac{\partial K}{\partial r}u+\frac{\partial K}{\partial s}v\right),\quad B(u,v)=3\sum_{i=0}^{2}\frac{2!}{i!(2-i)!}\frac{\partial^2 K}{\partial^{2-i}r\partial^i s}u^{2-i}v^i,$$
$$C(u,v)=\sum_{i=0}^{3}\frac{3!}{i!(3-i)!}\frac{\partial^3 K}{\partial^{3-i}r\partial^i s}u^{3-i}v^i.$$

Then

$$\mu=-B(u,v)\pm[B^2(u,v)-4A(u,v)C(u,v)]^{1/2},\ \text{and}\ \lambda=2A(u,v).$$

So a parameterization of $K(r,s,t)=0$ is

$$\mathbf{r}(u,v)=\{-B(u,v)\pm[B^2(u,v)-4A(u,v)C(u,v)]^{1/2}\}\mathbf{r}_0+2A(u,v)\mathbf{u} \tag{8}$$
$$=\mathbf{s}(u,v)\pm\mathbf{t}_0\sqrt{D(u,v)}, \tag{9}$$

where

$$\mathbf{s}(u,v)=-B(u,v)\mathbf{r}_0+2A(u,v)\mathbf{u},\ \ \mathbf{t}_0=\mathbf{r}_0,$$
$$D(u,v)=B^2(u,v)-4A(u,v)C(u,v). \tag{10}$$

Mapping $\mathbf{r}(u,v)$ by $\mathbf{P}(r,s,t)$ onto the quadric $F(\mathbf{X})=0$ as in Section 4.2.1, we get the (double valued) parameterization of the QSIC,

$$\tilde{\mathbf{Q}}(u,v)=\mathbf{r}(u,v)\mathbf{M}\mathbf{r}^{\mathrm{T}}(u,v) \tag{11}$$
$$=\mathbf{s}(u,v)\mathbf{M}\mathbf{s}^{\mathrm{T}}(u,v)+D(u,v)\mathbf{t}_0\mathbf{M}\mathbf{t}_0^{\mathrm{T}}\pm2\sqrt{D(u,v)}\mathbf{s}(u,v)\mathbf{M}\mathbf{t}_0^{\mathrm{T}} \tag{12}$$
$$=\tilde{\mathbf{S}}(u,v)\pm\tilde{\mathbf{T}}(u,v)\sqrt{D(u,v)}, \tag{13}$$

where

$$\tilde{\mathbf{S}}(u,v)=\mathbf{s}(u,v)\mathbf{M}\mathbf{s}^{\mathrm{T}}(u,v)+D(u,v)\mathbf{t}_0\mathbf{M}\mathbf{t}_0^{\mathrm{T}},\ \text{and}\ \tilde{\mathbf{T}}(u,v)=2\mathbf{s}(u,v)\mathbf{M}\mathbf{t}_0^{\mathrm{T}}.$$

By Theorem 3, $\mathbf{r}(u,v)$ passes through the base points $\mathbf{b}_0$ and $\mathbf{b}_1$ of $\mathbf{P}(r,s,t)$. Suppose $\mathbf{r}(u_i,v_i)=\mathbf{b}_i,\ i=0,1$. Then it is easy to see that $\tilde{\mathbf{Q}}(u_i,v_i)=\mathbf{P}(\mathbf{b}_i)=0$; therefore, as we shall see shortly in Lemma 4, the components of $\tilde{\mathbf{Q}}(u,v)$ have a common factor. This common factor can be removed by applying the following lemma.

Lemma 4. *In Eq.* (13), $\tilde{\mathbf{S}}(u,v)$ *and* $\tilde{\mathbf{T}}(u,v)$ *have a common factor* (v_0u-u_0v), *i.e.*, $\tilde{\mathbf{S}}(u_0,v_0)=\tilde{\mathbf{T}}(u_0,v_0)=0$, *where* (u_0,v_0) *is such that* $\mathbf{w}((u_0,v_0,0);\mu,\lambda)$ *coincides with the base line of* $\mathbf{P}(r,s,t)$.

Proof. Since $\mathbf{r}_0$ is on the base line $L(r,s,t)=0$ of $\mathbf{P}(r,s,t)$, there is a unique member $\mathbf{w}(\mathbf{u}_0;\mu,\lambda)$ in the pencil that coincides with $L(r,s,t)=0$. Let $\mathbf{u}_0=(u_0,v_0,0)$. Then $\mathbf{r}(u_0,v_0)$ (double valued, given by (9)) are the base points of $\mathbf{P}(r,s,t)$, which are not necessarily real points. Therefore $\tilde{\mathbf{Q}}(u_0,v_0)=0$, or equivalently,

$$\tilde{\mathbf{S}}(u_0,v_0)+\tilde{\mathbf{T}}(u_0,v_0)\sqrt{D(u_0,v_0)}=0,$$

$$\tilde{\mathbf{S}}(u_0,v_0)-\tilde{\mathbf{T}}(u_0,v_0)\sqrt{D(u_0,v_0)}=0.$$

It follows that $\tilde{\mathbf{S}}(u_0,v_0)=0$.

When $D(u_0,v_0)\neq 0$, obviously we have $\tilde{\mathbf{T}}(u_0,v_0)=0$. When $D(u_0,v_0)=0$, by (9), $\mathbf{r}(u_0,v_0)=\mathbf{s}(u_0,v_0)$ is a double base point of $\mathbf{P}(r,s,t)$. Thus the base line of $\mathbf{P}(r,s,t)$ is tangent at $\mathbf{s}(u_0,v_0)$ to the four conics $\mathbf{r}\mathbf{M}_i\mathbf{r}^{\mathrm{T}}=0$, $i=0,1,2,3$, formed by the four components of $\mathbf{P}(r,s,t)$ (see Section 2.2). Since $\mathbf{t}_0=\mathbf{r}_0$ is on the base line, it is on the tangent line to each of the four conics at the double base point $\mathbf{s}(u_0,v_0)$, i.e., $\mathbf{s}(u_0,v_0)\mathbf{M}_i\mathbf{t}_0^{\mathrm{T}}=0$. Hence $\tilde{\mathbf{T}}(u_0,v_0)=2\mathbf{s}(u_0,v_0)\mathbf{M}\mathbf{t}_0^{\mathrm{T}}=0$. □

By Lemma 4, after removing the common factor introduced by the base points, a parameterization of the QSIC is given by

$$\begin{aligned}\mathbf{Q}(u,v)&=\mathbf{S}(u,v)\pm\mathbf{T}(u,v)\sqrt{D(u,v)}s\\&\equiv\frac{\tilde{\mathbf{S}}(u,v)}{(v_0u-u_0v)}\pm\frac{\tilde{\mathbf{T}}(u,v)}{(v_0u-u_0v)}\sqrt{D(u,v)}.\end{aligned}\tag{14}$$

Here $\mathbf{S}(u,v)$ is of degree 3, $\mathbf{T}(u,v)$ is of degree 1, and $D(u,v)$ is of degree 4.

Adopting the convention that the degree of $\sqrt{D(u,v)}$ is half the degree of $D(u,v)$, the parameterizations constructed above for nonsingular irreducible QSICs have degree 3. As a comparison, the degree of the parameterizations derived in [19,40] is also 3, unless a cone is used as the parameterization surface which results in a parameterization of degree 4.

4.3. Connected components of a nonsingular QSIC

It is commonly accepted that a nonsingular QSIC has one or two topologically connected components in $\mathcal{PR}^3$; however, a proof of this fact in the literature is not known to us. Clearly, the number of connected components of a QSIC is important for its structural classification. To the best of our knowledge, none of the existing algorithms can identify these connected components. We will see that this issue is easy to resolve by exploiting the relationship between QSICs and cubic plane curves.

Theorem 5. *A nonsingular QSIC has one or two connected components in* $\mathcal{PR}^3$.

Proof. This theorem follows from the fact that the QSIC is a projection of a nonsingular plane cubic curve $K(r,s,t)=0$, which has one or two connected components in $\mathcal{PR}^2$ [29] (refer to Figs. 1(d)–(e)). □

Theorem 6. (1) *A nonsingular QSIC has one component in* $\mathcal{PR}^3$ *if and only if* $D(u, v)$ (*given by* (10)) *has two distinct real zeros*. (2) *A nonsingular QSIC has two components if and only if* $D(u, v) > 0$ *for all* $(u, v) \neq (0, 0)$ *or* $D(u, v)$ *has four distinct real zeros*.

Proof. Since this theorem is about a projective property, we may assume that the cubic curve $K(r, s, t) = 0$ is in the standard form (4).

Because $D(u, v)$ is the discriminant of the quadratic equation (7) in μ and λ, a pair (u_0, v_0) is a zero of $D(u, v)$ if and only if Eq. (7) has a double root, i.e., if and only if the corresponding line $\mathbf{w}((u_0, v_0, 0); \mu, \lambda)$ is tangent to the cubic curve $K(r, s, t) = 0$. The two limit intersection points forming the tangency are in general different from $\mathbf{r}_0$, the center of the pencil.

First suppose $K(r, s, t) = 0$ has one component in $\mathcal{PR}^2$ (see Fig. 1(e)). In this case, according to [29], from a point on $K(r, s, t) = 0$ two tangents can be drawn to $K(r, s, t) = 0$. So $D(u, v)$ has two real zeros. Furthermore, the two zeros are distinct; for otherwise, if $D(u, v)$ has a double zero, there would be a line passing through $\mathbf{r}_0$ that has a triple intersection point with $K(r, s, t) = 0$ besides $\mathbf{r}_0$, resulting in four intersection between the line and the cubic $K(r, s, t) = 0$, which contradicts Bézout's theorem. When the triple intersection point coincides with $\mathbf{r}_0$, the same argument still holds since the intersection multiplicity of the line and $K(r, s, t) = 0$ at $\mathbf{r}_0$ would be 4. Hence, in this case $D(u, v)$ has two distinct zeros.

Now suppose $K(r, s, t) = 0$ has two components in $\mathcal{PR}^2$, consisting of an oval and an infinite component (see Fig. 1(d)). When the pencil center $\mathbf{r}_0$ is on the oval, because the oval is strictly convex [29], any line passing through $\mathbf{r}_0$ intersects the oval at one more point, and intersects the infinite component at the third point. Recalling that $D(u, v)$ is the discriminant of (7), i.e., $D(u, v) > 0$ if and only if the associated line $\mathbf{w}((u, v, 0); \mu, \lambda)$ through $\mathbf{r}_0$ has two distinct real intersection points with $K(r, s, t) = 0$, we conclude $D(u, v) > 0$ for all $(u, v) \neq (0, 0)$.

When the pencil center $\mathbf{r}_0$ is on the infinite component, according to [29], from $\mathbf{r}_0$ on $K(r, s, t) = 0$ four tangents can be drawn to $K(r, s, t) = 0$, two to the oval, and the other two to the infinite component. So in this case there are four real zeros of $D(u, v)$. Using the same argument as in the case above where $K(r, s, t) = 0$ has one component, it can be shown that the four tangents drawn from $\mathbf{r}_0$ to $K(r, s, t) = 0$ are pairwise distinct. So the four real zeros of $D(u, v)$ are distinct.

Note that a nonsingular QSIC has one (two) component(s) in $\mathcal{PR}^3$ if and only if $K(r, s, t) = 0$ has one (two) component(s). Since, by Theorem 5, a nonsingular QSIC can only have one or two components, the theorem is proved. □

Now consider how each component of the QSIC is parameterized over the intervals of the real projective line $\mathcal{PR}^1$ divided by the zeros of $D(u, v)$. When the QSIC has one component, $\mathcal{PR}^1$ is divided into two intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ by the two distinct zeros of $D(u, v)$. Suppose $D(u, v)$ is positive over $\mathcal{I}_1$. Then the single component of the QSIC is parameterized by (14) over $\mathcal{I}_1$. The two ends of $\mathcal{I}_1$ correspond to the two tangents drawn from $\mathbf{r}_0$ to the only component of $K(r, s, t) = 0$.

When $K(r,s,t)=0$ has two components, either (1) $D(u,v)>0$ over the whole line $\mathcal{PR}^1$; or (2) $\mathcal{PR}^1$ is divided into four intervals $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, and $\mathcal{I}_4$ by the four distinct zeros of $D(u,v)$. In case (1), $\mathbf{r}_0$ is on the oval (see the proof of Theorem 6) and any line passing through $\mathbf{r}_0$ intersects $K(r,s,t)=0$ at two other distinct points: one on the oval and the other on the infinite component. Hence, when $D(u,v)$ has no real zeros, the two components of $K(r,s,t)=0$ (also the QSIC) are both parameterized by (14) over $\mathcal{PR}^1$ but distinguished by the + and − signs in front of $\sqrt{D(u,v)}$.

In case (2), $\mathbf{r}_0$ is on the infinite component. Without loss of generality, suppose that $D(u,v)>0$ over $\mathcal{I}_1$ and $\mathcal{I}_3$, and that a particular parameter $(u_0,v_0)\in\mathcal{I}_1$ gives rise to a line $\mathbf{w}((u_0,v_0,0);\mu,\lambda)$ through $\mathbf{r}_0$ that intersects the oval component of $K(r,s,t)=0$. Since the oval is strictly convex, it is intersected by this line at two points, which must be distinguished by the + and − signs in front of $\sqrt{D(u,v)}$. Hence the oval is traced over the interval $\mathcal{I}_1$, and the two branches given by the + and − signs cover the same oval. The two ends of $\mathcal{I}_1$ correspond to the two tangents drawn from $\mathbf{r}_0$ to the oval. Hence, the component of the QSIC that corresponds to the oval is covered by the + and − branches of the parameterization (14) over $\mathcal{I}_1$. Consequently, the other component of the QSIC corresponding to the infinite component of $K(r,s,t)=0$ is parameterized by the + and − branches of (14) over $\mathcal{I}_3$. Hence, when $D(u,v)$ has four real zeros, the two components of the QSIC are defined over two different intervals of $\mathcal{PR}^1$.

The analysis and solution of a quartic equation play a key role in the above discussion. Given a quartic equation, the existence of its real roots can be detected easily by Sturm's sequences. Also, the roots of a quartic equation can be solved using closed form formulas with radicals. See [11,37] for a detailed discussion of these techniques.

4.4. Accounting for generating lines contained in the QSIC

Detecting whether a generating line is contained in a QSIC is certainly necessary, but can prove difficult for some algorithms. For instance, this problem cannot be solved by Levin's method, as pointed out in [40]. In our algorithm, when the QSIC contains a generating line of $F(\mathbf{X})=0$ that passes through the COP $\mathbf{X}_0$, special treatment is required, since in this case the entire generating line is mapped by the inverse of $\mathbf{P}(r,s,t)$ into a base point. So this linear component of the QSIC cannot be obtained by mapping points on the quartic curve $\bar{G}(r,s,t)=0$ by $\mathbf{P}(r,s,t)$. Below we shall show how to detect this case by a simple analysis of the quartic curve $\bar{G}(r,s,t)=0$. Note that a linear component of the QSIC not passing through $\mathbf{X}_0$ can easily be detected by factoring $K(r,s,t)=0$.

Let $F(\mathbf{X})=0$ and $G(\mathbf{X})=0$ be two distinct quadrics, and let $\mathbf{P}(r,s,t)$ be a faithful rational quadratic parameterization of $F(\mathbf{X})=0$ with COP at $\mathbf{X}_0$. Let $L(r,s,t)=0$ be the base line of $\mathbf{P}(r,s,t)$.

Theorem 7. *Suppose that $F(\mathbf{X})=0$ is nondegenerate. Denote by ℓ the maximum integer such that $L(r,s,t)$ is an ℓ-fold factor of $\bar{G}(r,s,t)$. Let $\mathcal{G}_0$ be a generating line of*

$F(\mathbf{X}) = 0$ passing through $\mathbf{X}_0$, and let $\mathbf{b}_0$ be the base point of $\mathbf{P}(r, s, t)$ corresponding to $\mathcal{G}_0$. Let k denote the number of times that the curve $K(r, s, t) \equiv \bar{G}(r, s, t)/L^{\ell}(r, s, t) = 0$ passes through $\mathbf{b}_0$. Then $\mathcal{G}_0$ is contained in the QSIC $k + \ell - 2$ times, with $0 \leqslant k + \ell - 2 \leqslant 2$.

Proof. By Theorem 2, ℓ is the multiplicity of the singularity of the QSIC at $\mathbf{X}_0$, and $1 \leqslant \ell \leqslant 4$ since the COP $\mathbf{X}_0$ is chosen to be on the QSIC. Let $\mathcal{D}$ denote the collection of all the remaining components of the QSIC besides $\mathcal{G}_0$. First assume that $\mathcal{G}_0$ is not tangent to $\mathcal{D}$ at $\mathbf{X}_0$. It is shown in [32] that k is equal to the number of those intersections of $\mathcal{G}_0$ with the components in $\mathcal{D}$ that are not at $\mathbf{X}_0$, with $1 \leqslant k \leqslant 3$. Let m be the multiplicity in the QSIC of the generating line $\mathcal{G}_0$. Let h be the multiplicity of the singularity of $\mathcal{D}$ at $\mathbf{X}_0$. Then h is also the multiplicity of intersections of $\mathcal{G}_0$ with $\mathcal{D}$ at $\mathbf{X}_0$. Since the multiplicity of the singularity of the QSIC at $\mathbf{X}_0$ is $\ell = h + m$, it follows that $h = \ell - m$. Hence the total number of intersections between $\mathcal{G}_0$ and $\mathcal{D}$ is $k + h = k + \ell - m$.

When $\mathcal{G}_0$ is a tangent to $\mathcal{D}$ at $\mathbf{X}_0$, the number of intersections of $\mathcal{G}_0$ with $\mathcal{D}$ that are not at $\mathbf{X}_0$ is $k - 1$ and the number of intersections of $\mathcal{G}_0$ with $\mathcal{D}$ at $\mathbf{X}_0$ is $h + 1$. Still, the total number of intersections between $\mathcal{G}_0$ and $\mathcal{D}$ is $(k - 1) + (h + 1) = k + h = k + \ell - m$.

Next we show that $k + h = 2$. Let d be the degree of $\mathcal{D}$ which is defined to be the sum of the degrees of all the components in $\mathcal{D}$. Then $d = 4 - m$. Let $\mathcal{G}_1$ be a generating line of $F(\mathbf{X}) = 0$ that intersects $\mathcal{G}_0$ and is not contained in the quadric $G(\mathbf{X}) = 0$. Let $\mathcal{P}$ denote the plane determined by $\mathcal{G}_0$ and $\mathcal{G}_1$. Then there are $4 - m$ intersections between $\mathcal{P}$ and $\mathcal{D}$, and these intersections are on either the line $\mathcal{G}_0$ or $\mathcal{G}_1$, since $\mathcal{G}_0$ and $\mathcal{G}_1$ are the only intersection between $\mathcal{P}$ and $F(\mathbf{X}) = 0$. Since the line $\mathcal{G}_1$ is not contained in the QSIC, it is not contained in $G(\mathbf{X}) = 0$, so it has two intersections with $G(\mathbf{X}) = 0$, or equivalently, two intersections with the QSIC. Since $\mathcal{G}_1$ intersects $\mathcal{G}_0$ with multiplicity m, the number of intersections of $\mathcal{G}_1$ with $\mathcal{D}$ is $2 - m$. So the number of intersections of $\mathcal{G}_0$ with $\mathcal{D}$ is $(4 - m) - (2 - m) = 2$. Hence $k + \ell - m = k + h = 2$, or $m = k + \ell - 2$.

Now we shall show that $0 \leqslant k + \ell - 2 \leqslant 2$. Although $k + \ell - 2 \geqslant 0$ is obvious geometrically from the fact that $\mathcal{G}_0$ is either contained $(k + \ell - 2 > 0)$ or not contained $(k + \ell - 2 = 0)$ in the QSIC, an algebraic argument is still provided as follows. It is already known that $1 \leqslant \ell \leqslant 4$. If $\ell \geqslant 2$, clearly $k + \ell - 2 \geqslant 0$, since $k \geqslant 0$. When $\ell = 1$, by Theorem 2, the COP $\mathbf{X}_0$ is a regular point of the QSIC. So, by Theorem 3, $k \geqslant 1$. Hence again we have $k + \ell - 2 \geqslant 0$.

Next we show by contradiction that $k + \ell - 2 \leqslant 2$. Suppose $k + \ell - 2 > 2$, i.e., $\mathcal{G}_0$ is contained in the QSIC at least three times. Then the QSIC consists of four lines. It therefore follows [12] that there exists a quadric consisting of two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ in the pencil spanned by $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$, and that $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect $F(\mathbf{X}) = 0$ along the same QSIC, i.e., the four lines. However, each of $\mathcal{P}_1$ and $\mathcal{P}_2$ can only intersect the nondegenerate quadric $F(\mathbf{X}) = 0$ along two intersecting and distinct generating lines. This contradicts the fact that $\mathcal{G}_0$ is contained in the QSIC at least three times. Hence $k + \ell - 2 \leqslant 2$. $\square$

Theorem 8. *Suppose that $F(\mathbf{X}) = 0$ is properly degenerate. Denote by ℓ the maximum integer such that $L(r,s,t)$ is an ℓ-fold factor of $\bar{G}(r,s,t)$. Let $\mathcal{G}_0$ be the unique generating line of $F(\mathbf{X}) = 0$ that passes through $\mathbf{X}_0$. Let $\mathbf{b}_0$ be the double base point of $\mathbf{P}(r,s,t)$, and let k denote the number of times that the curve $K(r,s,t) \equiv \bar{G}(r,s,t)/L^\ell(r,s,t) = 0$ passes through $\mathbf{b}_0$. Then $\mathcal{G}_0$ is contained in the QSIC $2\ell + k - 4$ times, with $0 < 2\ell + k - 4 \leqslant 4$.*

Proof. The degree of $K(r,s,t) = 0$ is $4 - \ell$. By Theorem 1, $K(r,s,t) = 0$ is mapped by $\mathbf{P}(r,s,t)$ into a component, or a collection of components, of the QSIC of total degree $d = 2 \times (4 - \ell) - k = 8 - 2\ell - k$. Because the QSIC is a quartic curve, $\mathcal{G}_0$ must be included in the QSIC $4 - d = 2\ell + k - 4$ times in order to account for the missing components of the QSIC.

It is again obvious geometrically that $2\ell + k - 4 \geqslant 0$, and an algebraic argument similar to the proof of Theorem 7 can be given but is omitted. Because $k \leqslant 4 - \ell$, $2\ell + k - 4 \leqslant \ell \leqslant 4$. □

It is possible for a generating line $\mathcal{G}_0$ of a properly degenerate quadric $F(\mathbf{X}) = 0$ (a cone or a cylinder) to be contained in the QSIC four times, as exemplified by the intersection of a cylinder $F(\mathbf{X}) = 0$ with a double plane tangent to $F(\mathbf{X}) = 0$ along $\mathcal{G}_0$. So the upper bound 4 is attainable.

5. Examples

In this section we shall use some examples to demonstrate the major steps of our algorithm in a variety of cases.

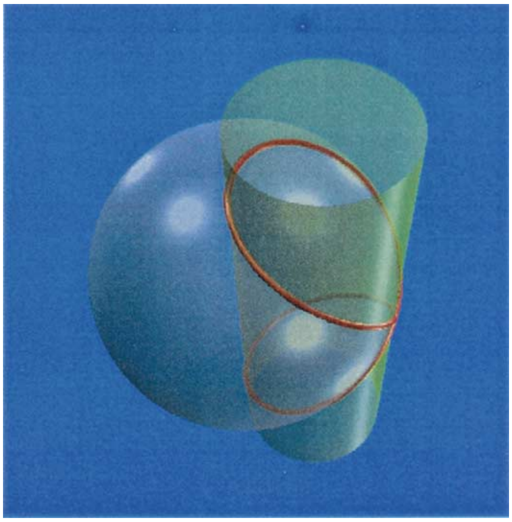

Fig. 2. Singular intersection of a sphere and a cylinder.

5.1. Example 1

Consider the sphere $S_1 : F(\mathbf{X}) \equiv x^2 + y^2 + z^2 - w^2 = 0$ and the cylinder $S_2 : G(\mathbf{X}) \equiv x^2 + (y - 0.5w)^2 - 0.25w^2 = 0$ illustrated in Fig. 2. Rewriting these equations in matrix form, we have

$$F(\mathbf{X}) = \mathbf{X}\mathbf{A}\mathbf{X}^{\mathrm{T}}, \quad G(\mathbf{X}) = \mathbf{X}\mathbf{B}\mathbf{X}^{\mathrm{T}},$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \end{bmatrix}.$$

A rational quadratic parameterization of $\mathcal{S}_1$ with COP at $X_0 = (0, 0, 1, 1)$ is given by

$$\mathbf{P}(r, s, t) = r^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + rs \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + rt \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + rs \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + st \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$
$$+ rt \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + st \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

The quartic curve $\bar{G}(r, s, t) \equiv G(\mathbf{P}(r, s, t)) = 0$ is

$$\bar{G}(r, s, t) = 2r^2st + 4r^2t^2 + 2s^3t + 4r^2t^2 + 2st^3 = 0.$$

Removing the base line factor $\mathbf{X}_0\mathbf{A}(r, s, t, 0)^{\mathrm{T}} = t$ from $\bar{G}(r, s, t) = 0$, we obtain the cubic curve

$$K(r, s, t) = 2r^2s + 4r^2t + 2s^3 + 4r^2t + 2st^2 = 0,$$

which is irreducible and has a singular point at $\mathbf{r}_0 = (0, -1, 1)$. Using the pencil of lines

$$\mathbf{P}((0, u, v); \mu, \lambda) = \mu\mathbf{r}_0 + \lambda(0, u, v),$$

centered at $\mathbf{r}_0$, we obtain a parametrization of $K(r, s, t) = 0$,

$$\begin{aligned} r(u, v) &= C(u, v)\mathbf{r}_0 - B(u, v)(u, v, 0) \\ &= u^3 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + u^2v \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + uv^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v^3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \end{aligned}$$

where $B(u,v) = u^2 - v^2$ and $C(u,v) = u^2v + v^3$. Mapping $\mathbf{r}(u,v)$ onto the sphere $\mathcal{S}_1$ yields a degree 6 rational parameterization of the singular QSIC,

$$\begin{aligned}\mathbf{P}(\mathbf{r}(u,v)) &= \mathbf{r}(u,v)\mathbf{M}\mathbf{r}^{\mathrm{T}}(u,v)\\ &= u^6\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + u^5v\begin{pmatrix}2\\0\\0\\0\end{pmatrix} + u^4v^2\begin{pmatrix}0\\4\\1\\3\end{pmatrix} + u^3v^3\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\\ &\quad + u^2v^4\begin{pmatrix}0\\4\\-1\\3\end{pmatrix} + uv^5\begin{pmatrix}-2\\0\\0\\0\end{pmatrix} + v^6\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}.\end{aligned} \tag{15}$$

By Theorem 3, $\mathbf{r}(u,v)$ passes once through each of the two base points $(1,\pm\mathrm{i},0)$ of $\mathbf{P}(r,s,t)$. The two parameter values of $\mathbf{r}(u,v)$ that correspond to the base points are determined by two of the roots of the cubic equation $\mathbf{1}_0\mathbf{r}^{\mathrm{T}}(r,s,t) = 0$, where $\mathbf{1}_0\mathbf{X}^{\mathrm{T}} = 0$ is the base line equation. Since $\mathbf{1}_0 = (0,0,1)$, $\mathbf{1}_0\mathbf{r}^{\mathrm{T}}(u,v) = u^2v + v^3 = v(u^2+v^2)$. The factor $u^2 + v^2$ has complex conjugate roots, so these two roots must give the two base points through $\mathbf{r}(u,v)$. By removing the factor $u^2 + v^2$ from $\mathbf{P}(\mathbf{r}(u,v))$ (15), we obtain a faithful quartic rational parameterization of the QSIC,

$$\begin{aligned}\bar{\mathbf{q}}(u,v) &= \frac{\mathbf{r}(u,v)}{u^2+v^2}\\ &= u^4\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + u^3v\begin{pmatrix}2\\0\\0\\0\end{pmatrix} + u^2v^2\begin{pmatrix}0\\4\\0\\2\end{pmatrix} + uv^3\begin{pmatrix}-2\\0\\0\\0\end{pmatrix} + v^4\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}.\end{aligned}$$

The singular point on the QSIC is at $\mathbf{P}(\mathbf{r}_0) = (1,0,1,1)$, which is a crunode since the discriminant of $B(u,v)$ is positive.

5.2. Example 2

Consider the sphere $\mathcal{S}_1 : F(\mathbf{X}) = x^2 + y^2 + z^2 - w^2 = 0$ and the cylinder $\mathcal{S}_2 : G(\mathbf{X}) = (x - 0.65w)^2 + y^2 - 0.4225w^2 = 0$ illustrated in Fig. 3. Rewriting these equations in matrix form, we have

$$F(\mathbf{X}) = \mathbf{XAX}^{\mathrm{T}}, \quad G(\mathbf{X}) = \mathbf{XBX}^{\mathrm{T}},$$

where

$$\mathbf{A} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}1&0&0&-0.65\\0&1&0&0\\0&0&0&0\\-0.65&0&0&0\end{bmatrix}.$$

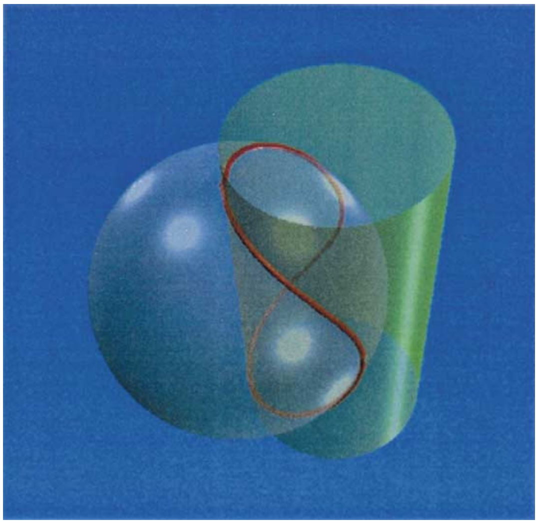

Fig. 3. Nonsingular intersection of a sphere and a cylinder.

A rational quadratic parameterization of $\mathcal{S}_1$ with COP at $\mathbf{X}_0 = (0, 0, 1, 1)$ is given by $\mathbf{P}(r, s, t)$, the same as in Example 1. The quartic curve $\bar{G}(r, s, t) \equiv G(\mathbf{P}(r, s, t)) = 0$ is

$$\bar{G}(r, s, t) = 2.6r^2st + 4.0r^2t^2 + 2.6s^3t + 4.0s^2t^2 + 2.6st^3 = 0.$$

Removing the base line factor $\mathbf{X}_0\mathbf{A}(r, s, t, 0)^{\mathrm{T}} = t$ from $\bar{G}(r, s, t) = 0$, we obtain the cubic curve

$$K(r, s, t) = 2.6r^2s + 4.0r^2t + 2.6s^3 + 4s^2t + 2.6st^2 = 0,$$

which is nonsingular. Its parameterization is

$$\mathbf{r}(u, v) = \mathbf{s}(u, v) \pm \mathbf{t}_0\sqrt{D(u, v)},$$

where $\mathbf{t}_0 = (1, 0, 0)^{\mathrm{T}}$,

$$\mathbf{s}(u, v) = u^2\begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} + uv\begin{pmatrix} 0 \\ 8 \\ 5.2 \end{pmatrix} + v^2\begin{pmatrix} 0 \\ 5.2 \\ 0 \end{pmatrix},$$

and

$$D(u, v) = -41.6u^3v - 91.04u^2v^2 - 83.2uv^3 - 27.04v^4.$$

Mapping $\mathbf{r}(u, v)$ onto the sphere $\mathcal{S}_1$ and eliminating the unfaithfulness introduced by the base points, yields an exact parameterization of the nonsingular QSIC

$$\mathbf{Q}(u, v) = \mathbf{S}(u, v) \pm \mathbf{T}(u, v)\sqrt{D(u, v)},$$

where $D(u, v)$ is the same as above, and

$$\mathbf{S}(u, v) = u^3 \begin{pmatrix} 0.0 \\ 0.0 \\ -64.0 \\ 64.0 \end{pmatrix} + u^2 v \begin{pmatrix} 0.0 \\ -128.0 \\ -124.8 \\ 41.6 \end{pmatrix} + uv^2 \begin{pmatrix} 0.0 \\ -166.4 \\ -54.08 \\ 0.0 \end{pmatrix} + v^3 \begin{pmatrix} 0.0 \\ -54.08 \\ 0.0 \\ 0.0 \end{pmatrix},$$

$$\mathbf{T}(u, v) = u \begin{pmatrix} -16.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix} + v \begin{pmatrix} -10.4 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}.$$

Since $D(u, v)$ has two real zeroes $(u_1, v_1) = (1.0, 0.0)$ and $(u_2, v_2) = (-0.65, 1.0)$, by Theorem 6 the QSIC has one component.

5.3. Example 3

Consider the two quadric cones $\mathcal{S}_1$ and $\mathcal{S}_2$ given by

$$F(\mathbf{X}) = \mathbf{X}\mathbf{A}\mathbf{X}^{\mathrm{T}}, \quad G(\mathbf{X}) = \mathbf{X}\mathbf{B}\mathbf{X}^{\mathrm{T}},$$

where

$$\mathbf{A} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.50 \\ 0.0 & 0.75 & -0.5 & -0.5 \\ 0.0 & -0.5 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 0.25 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.75 & 0.0 & -0.5 & 0.125 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.0 & 0.25 \\ 0.125 & 0.0 & 0.25 & -0.3125 \end{bmatrix}.$$

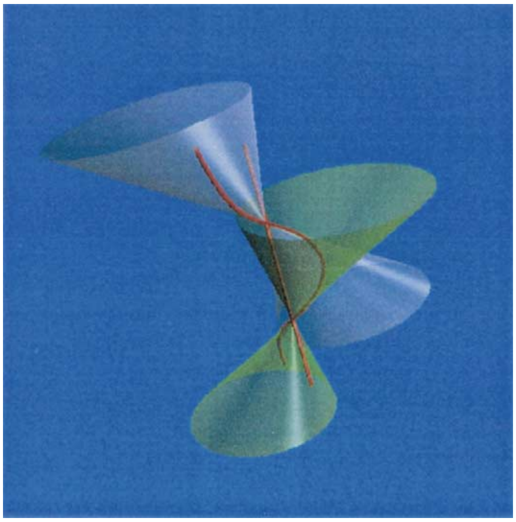

Fig. 4. Singular intersection of two cones.

This example is illustrated in Fig. 4. The vertex of $\mathcal{S}_1$ is on $\mathcal{S}_2$ and the vertex of $\mathcal{S}_2$ is on $\mathcal{S}_1$. A rational quadratic parameterization of $\mathcal{S}_1$ with COP at $\mathbf{X}_0 = (0.5, 0.0, 0.0, 1.0)$ is given by

$$\mathbf{P}(r,s,t) = r^2\begin{pmatrix}0.5\\0.0\\0.0\\1.0\end{pmatrix} + rs\begin{pmatrix}0.5\\0.0\\0.0\\0.0\end{pmatrix} + rt\begin{pmatrix}0.0\\0.0\\0.0\\0.0\end{pmatrix} + rs\begin{pmatrix}0.5\\0.0\\0.0\\0.0\end{pmatrix} + s^2\begin{pmatrix}0.375\\1.0\\0.0\\0.75\end{pmatrix}$$
$$+ st\begin{pmatrix}-0.25\\0.0\\0.5\\-0.5\end{pmatrix} + rt\begin{pmatrix}0.0\\0.0\\0.0\\0.0\end{pmatrix} + st\begin{pmatrix}-0.25\\0.0\\0.5\\-0.5\end{pmatrix} + t^2\begin{pmatrix}0.0\\0.0\\0.0\\0.0\end{pmatrix}.$$

The quartic curve corresponding to the QSIC is

$$\bar{G}(r,s,t) = r^3 s + 0.75r^2s^2 + 0.75rs^3 - 2rs^2t + s^4 = 0.$$

$\mathbf{P}(r,s,t)$ has the double base point $(0,0,1)$. The base line is

$$L(r,s,t) = \mathbf{l}_0(r,s,t)^{\mathrm{T}} = \mathbf{X}_0\mathbf{A}(r,s,t,0)^{\mathrm{T}} = -0.5s = 0.$$

Dividing $\bar{G}(r,s,t)$ by $L(r,s,t)$ yields the cubic curve

$$K(r,s,t) = -2.0r^3 - 1.5r^2s - 1.5rs^2 + 4.0rst - 2.0s^3 = 0,$$

which is singular, with the singular point at $(r_0, s_0, t_0) = (0,0,1)$. A rational cubic parameterization of $K(r,s,t)$ is found to be

$$\mathbf{r}(u,v) = C(u,v)\mathbf{r}_0 - B(u,v)(u,v,0) = u^3\begin{pmatrix}0.0\\0.0\\-2.0\end{pmatrix} + u^2v\begin{pmatrix}-4.0\\0.0\\-1.5\end{pmatrix}$$
$$+ uv^2\begin{pmatrix}0.0\\-4.0\\-1.5\end{pmatrix} + v^3\begin{pmatrix}0.0\\0.0\\-2.0\end{pmatrix}$$

Mapping $\mathbf{r}(u,v)$ onto $\mathcal{S}_1$ by $\mathbf{P}(r,s,t)$ yields a parameterization of the QSIC,

$$\begin{aligned}\mathbf{P}(\mathbf{r}(u,v)) &= \mathbf{r}(u,v)\mathbf{M}\mathbf{r}^{\mathrm{T}}(u,v)\\ &= u^6\begin{pmatrix}0.0\\0.0\\0.0\\0.0\end{pmatrix} + u^5v\begin{pmatrix}0.0\\0.0\\0.0\\0.0\end{pmatrix} + u^4v^2\begin{pmatrix}4.0\\0.0\\8.0\\8.0\end{pmatrix} + u^3v^3\begin{pmatrix}13.0\\0.0\\6.0\\-6.0\end{pmatrix}\\ &\quad + u^2v^4\begin{pmatrix}3.0\\16.0\\6.0\\6.0\end{pmatrix} + uv^5\begin{pmatrix}-4.0\\0.0\\8.0\\-8.0\end{pmatrix} + v^6\begin{pmatrix}0.0\\0.0\\0.0\\0.0\end{pmatrix}.\end{aligned}$$

To remove unfaithfulness, we observe that the cubic $\mathbf{r}(u,v)$ passes through the double base point $(0,0,1)$ three times. The three intersections of $\mathbf{r}(u,v)$ with the base line $\mathbf{l}_0^{\mathrm{T}}(r,s,t)=0$ correspond to the three roots of $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u,v)=0$. So, removing the cubic factor $\mathbf{l}_0\mathbf{r}^{\mathrm{T}}(u,v)=2uv^2$ from $\mathbf{P}(\mathbf{r}(u,v))$, we obtain a rational parameterization of the cubic component of the QSIC,

$$\mathbf{Q}(u,v)=u^3\begin{pmatrix}2.0\\0.0\\4.0\\4.0\end{pmatrix}+u^2v\begin{pmatrix}6.5\\0.0\\3.0\\-3.0\end{pmatrix}+uv^2\begin{pmatrix}1.5\\8.0\\3.0\\3.0\end{pmatrix}+v^3\begin{pmatrix}-2.0\\0.0\\4.0\\-4.0\end{pmatrix}.$$

Since $L(r,s,t)=\mathbf{l}_0(r,s,t)^{\mathrm{T}}$ is a single factor of $\bar{G}(r,s,t)$ and $K(r,s,t)\equiv\bar{G}(r,s,t)/L(r,s,t)=0$ passes through the base point $(0,0,1)$ three times, it follows by Theorem 8, with $\ell=1$ and $k=3$, that the unique generating line of $\mathcal{S}_1$ passing through $\mathbf{X}_0$ is contained once in the QSIC, since $2\ell+k-4=1$. This line is parameterized by

$$f\mathbf{Q}_1(u,v)=u\begin{pmatrix}0.5\\0.0\\0.0\\1.0\end{pmatrix}+v\begin{pmatrix}0.0\\0.0\\0.5\\0.0\end{pmatrix}.$$

Thus the QSIC consists of a cubic curve and a line. Note that, since the singular point of $K(r,s,t)=0$ is at the base point of $\mathbf{P}(r,s,t)$, it is not mapped to any point on the QSIC.

5.4. Example 4

Consider the elliptical cylinder $\mathcal{S}_1: F(\mathbf{X})=4x^2+z^2-w^2=0$ and the hyperboloid of one sheet $\mathcal{S}_2: G(\mathbf{X})=x^2+4y^2-z^2-w^2=0$ illustrated in Fig. 5. Only four digits after the decimal point are kept in the following presentation. The COP used for parameterizing $\mathcal{S}_1$ is $\mathbf{X}_0=(0,\sqrt{1/2},1,1)$. The QSIC is found to be nonsingular. After removing unfaithfulness, the parameterization of the QSIC is

$$\mathbf{Q}(u,v)=\mathbf{S}(u,v)\pm\mathbf{T}(u,v)\sqrt{D(u,v)},$$

where

$$\mathbf{S}(u,v)=u^3\begin{pmatrix}0.0\\1131.3708\\-1600.0\\1600.0\end{pmatrix}+u^2v\begin{pmatrix}0.0\\-5760.0\\10861.1602\\3620.2867\end{pmatrix}+uv^2\begin{pmatrix}0.0\\10861.1602\\-21504.0\\5120.0\end{pmatrix}$$
$$+v^3\begin{pmatrix}0.0\\-8192.0\\11585.2375\\11585.2375\end{pmatrix},$$

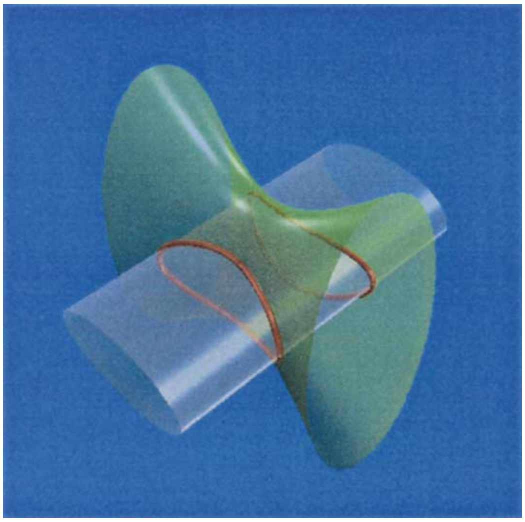

Fig. 5. Nonsingular intersection of an elliptical cylinder and a hyperboloid of one sheet.

$$\mathbf{T}(u,v) = u\begin{pmatrix} -80.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix} + v\begin{pmatrix} 1181.0193 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$$

and

$$D(u,v) = 905.0967u^3v - 3328.0u^2v^2 + 2896.3094uv^3.$$

Since $D(u,v)$ has four real zeroes, by Theorem 6 the QSIC has two connected components.The computation error in $\mathbf{Q}(u,v)$ is of order $O(10^{-7})$ with double precision. The error is measured as the maximum distance from a sequence of densely sampled points on $\mathbf{Q}(u,v)$ to the two original quadrics.

5.5. Example 5

Consider the ellipsoid

$$\mathcal{S}_1 : F(\mathbf{X}) \equiv 0.95x^2 + 1.1\, y^2 + 1.05z^2 - w^2 = 0$$

and the sphere

$$\mathcal{S}_2 : G(\mathbf{X}) \equiv x^2 + y^2 + z^2 - w^2 = 0$$

illustrated in Fig. 6. The COP for parameterizing $\mathcal{S}_1$ is at $\mathbf{X}_0 = (\sqrt{1/2}, 0, \sqrt{1/2}, 1)$. After removing unfaithfulness, a parameterization of the QSIC is

$$\mathbf{Q}(u,v) = \mathbf{S}(u,v) \pm \mathbf{T}(u,v)\sqrt{D(u,v)}, \tag{16}$$

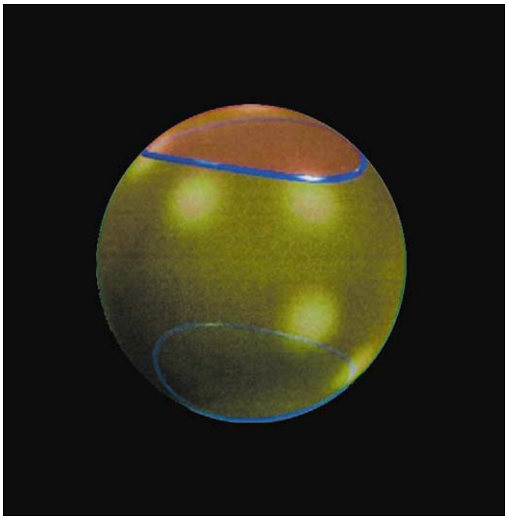

Fig. 6. Singular intersection of a sphere (red) and an ellipsoid (green).

where

$$\mathbf{S}(u,v) = u^3\begin{pmatrix} -0.72 \\ 0.0 \\ 0.72 \\ 1.0182 \end{pmatrix} + u^2v\begin{pmatrix} -0.72 \\ 0.0 \\ -1.2 \\ 0.3394 \end{pmatrix} + uv^2\begin{pmatrix} 0.08 \\ 0.0 \\ -0.72 \\ 0.3394 \end{pmatrix} + v^3\begin{pmatrix} 0.08 \\ 0.0 \\ -0.08 \\ 0.1131 \end{pmatrix},$$

$$\mathbf{T}(u,v) = u\begin{pmatrix} 0.0 \\ 1.6972 \\ 0.0 \\ 0.0 \end{pmatrix} + v\begin{pmatrix} 0.0 \\ 0.5656 \\ 0.0 \\ 0.0 \end{pmatrix},$$

$$D(u,v) = 0.48u^3v - 0.32u^2v^2 - 0.16uv^3.$$

Since $D(u,v)$ has four real zeroes $(u_1,v_1) = (1.0, 0.0)$, $(u_2,v_2) = (1.0, 1.0)$, $(u_3,v_3) = (-0.3333, 1.0)$, $(u_4,v_4) = (0.0, 1.0)$, it follows by Theorem 6 that the QSIC has two connected components.

Using double precision, the computation error in this parameterization is less than 10^{-14}, and the QSIC is correctly classified as nonsingular. Again, the error is measured as the maximum distance from a sequence of densely sampled points on the parametric curve (16) to the two input quadrics. It is interesting to note that when the Segre characteristic is applied to this input, the discriminant of the characteristic equation of the two quadrics evaluates to less than 10^{-13}. We speculate that in this case an erroneous classification of the QSIC is very likely to result if the discriminant is evaluated with floating-point arithmetic.

6. Concluding remarks

We have presented an algebraic method to classify and parameterize the intersection curve of two quadric surfaces (QSIC). This method is different from those based on Levin's approach [19,31,40], and is an extension of the method in [12]. Based on a birational mapping between the QSIC and a plane cubic curve, we have developed an algorithm to classify a general QSIC and compute parameterizations of all its irreducible or connected components. One requirement of the method is that a real point on the QSIC be found first and used as the COP for a rational quadratic parameterization of one of the input quadrics. In our implementation this point is computed by Levin's method.

Singular points on boundary curves furnish important topological information in solid modeling systems [17]. By exploiting the relation between QSICs and plane cubic curves, the singular point on a singular irreducible QSIC can easily be classified as corresponding to the crunodal, acnodal, or cuspidal double point on a singular plane cubic curve. Moreover, based on the analysis of plane cubic curves, our method is capable of identifying the connected components of a nonsingular QSIC in $\mathcal{PR}^3$, corresponding to either a unipartite or bipartite nonsingular plane cubic curve [29]. These connected components cannot be identified by other established methods for computing QSICs.

To summarize, compared with other existing algorithms, the main advantages of our method are that (1) it accepts arbitrary quadric surfaces as input (all methods based on geometric approaches deal with natural quadrics only); (2) it computes a general intersection curve between any two quadrics, degenerate as well as nondegenerate (the method in [12], for example, detects and processes degenerate intersection curves only); (3) it produces a complete topological classification of the intersection curve of two quadric surfaces in terms of singularity type and the number of connected components, a classification not achieved in any of the other existing methods; the method in [40] can detect the degeneracy of a QSIC and determine the type of its singularity, but cannot determine the number of connected components of a nondegenerate QSIC.

The birational mapping between the QSIC and a plane cubic curve is well known in the literature of algebraic geometry. Our contribution is exploiting this result to devise a new algorithm to compute the intersection curve between two general quadric surfaces. In this endeavor we have had to solve a number of computational problems and to provide a thorough analysis of all nondegenerate and degenerate cases encountered in practice. As a result, we have obtained an algorithm capable of accepting general quadrics, producing a complete topological classification of the QSIC, and computing a low-degree parameterization of a general QSIC. In these aspects our algorithm is superior to other existing methods of computing the QSIC.

It is still an open problem of theoretical interest to determine whether or not every plane cubic curve can be generated from the projection of a QSIC through a point on the QSIC. So far it has been observed only that all major species of cubic plane curves occur as the projections of QSICs in our algorithm. Therefore our efforts to process a general plane cubic curve are justified. To put this problem in perspective, recall that only special plane quartic curves can occur as the projections of

QSICs through an arbitrary point in space; for a nonsingular QSIC such a quartic has, in general, two double points. So another open problem is how to exploit the properties of such a special quartic curve to facilitate the computation of a QSIC.

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